State Mathematics Finals: Level II

May 3, 2018

1. Consider the sequence $\{a_n\}$ that is defined recursively by
$a_1 = 10, \quad a_{n+1} = \sqrt{a_n}$ for $n \geq 1$.

If the sequence is continued indefinitely, which of the following numbers gives the limit of $\{a_n\}$?

(A) $2\sqrt{5}$ (B) $\sqrt{21}$ (C) 3 (D) 5 (E) 1

2. Three athletes run a 1500 meter- race. When the first runner finishes, the second runner and third runner are at 100 and 170 meters from the finish line, respectively. If each runner maintains a constant speed, how far is the third runner from the finish line when the second runner finishes?

(A) $70\ m$ (B) $72.5\ m$ (C) $74.5\ m$ (D) $75\ m$ (E) $77\ m$

3. A virulent infectious disease spreads as given by the equation $N(t) = N_0(1 - 10^{-kt})$ where:
t is time in hours,
$N(t)$ is number of people infected at time t
N_0 is the total population of the community at time $t = 0$
k is the infection rate constant of the disease $(k = 0.001)$.

Calculate how long will it take for 90% of the population to be infected.

(A) $90\ hours$ (B) $100\ hours$ (C) $1000\ hours$ (D) $990\ \ hours$ (E) None of these

4. Simplify the expression:

$$\left(\frac{1}{1+\sqrt[3]{2}+\sqrt[3]{4}}\right)^3$$

(A) $1 + 3\sqrt[3]{2} - 3\sqrt[3]{4}$ (B) $1 - \sqrt[3]{2} - \sqrt[3]{4}$ (C) $1 - 3\sqrt[3]{2} + 3\sqrt[3]{4}$ (D) $\sqrt[3]{2} - 1$
(E) None of these

5. Consider the sequence $\{ a_n\}$ such that $a_0 = 4,\ a_1 = 2,\ a_n = a_{n-1} - a_{n-2}$. Compute the sum: $a_0 + a_1 + a_2 + \cdots + a_{101}$.

(A) 4 (B) 2 (C) -2 (D) -4 (E) 0

6. Consider the triangle *ABC* with vertices with coordinates: $A = (4, -3)$, $B = (4, 1)$, and $C = (2,1)$. Calculate the area of the triangle *ABC*.

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) 4 (E) None of these

7. Let *x* and *y* be real variables such that $x \geq 0$ and $y \geq 0$. Suppose $f(x, y) = (x^2 + y^2 - 2)^2 + 4(xy + 2)(x^2 + xy + y^2)$. Compute $\sqrt{f(x, y)}$.

(A) $x^2 + y^2$ (B) $(x^2 + y^2 - 2)^{1/2} + 2xy$ (C) $2(xy + 2)^2$ (D) $(x + y)^2 + 2$ (E) $4xy$

8. Consider the set $S = \{a_1,\ a_2,\ a_3, \ldots\ a_n\ \}$. Let the arithmetic mean of *S* be denoted by $E(S)$ and the variance of *S* be given by $V(S)$ such that:

$$E(S) = \frac{1}{n}\sum a_i$$

$$V(S) = \frac{1}{n-1}\sum(a_i - E(S))^2$$

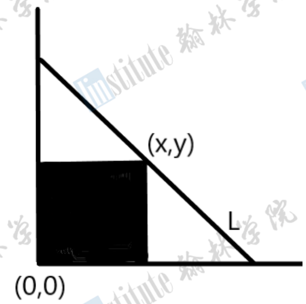
If each element of *S* is multiplied by *k* and then increased by *l*, find the value of the new variance of the set *S*.

(A) $E(S) + kl$ (B) $k^2V(S)$ (C) $k^2V(S) + l$ (D) $l^2V(S) + k$ (E) $(E(S))^2 - k^2$

9. In the mythical city of Jamais, the 6-faced die used in *Casino Mathematicale* has the face numbers $\{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$ handcrafted such that the probability of a number *x* appearing during a random toss is proportional to the number *x*. Compute the probability that when a single die is tossed randomly, an odd prime number is observed.

(A) $\frac{2}{9}$ (B) $\frac{1}{7}$ (C) $\frac{8}{21}$ (D) $\frac{3}{4}$ (E) None of these

10. Consider the shaded rectangle in the figure shown.



If the equation of L is $2x + y = 100$, find the value of the maximum possible area of the shaded rectangle.

(A) 1150 $unit^2$ (B) 1250 $unit^2$ (C) 1300 $unit^2$ (D) 1350 $unit^2$ (E) None of these

11. Consider the ordered triple $\{a_1,\ a_2,\ a_3\}$ such that $\frac{1}{1+\sqrt{2}+\sqrt{3}} = a_1 + a_2\sqrt{2} + a_3\sqrt{6}$. Find the sum of a_1, a_2, and a_3.

(A) 0 (B) $\frac{1}{6}$ (C) $\frac{1}{3}$ (D) $\frac{3}{8}$ (E) $\frac{1}{2}$

12. Suppose a, b, c, and d are non-zero real numbers. If $ax^2 + bx + c = 0$ has two real solutions of opposite signs and $ax^2 + bx + c = d$ has two real solutions of the same sign, which of the following statements is true?

(A) $ac < 0,\ |c| < |d|$ (B) $ac < 0,\ |c| > |d|$ (C) $ac > 0,\ |c| < |d|$

(D) $ac > 0,\ |c| > |d|$ (E) $ac > 0,\ c > d$

13. Let x be a real number. Consider the equation $x^{1/2} - 9x^{1/3} + 20x^{1/6} = 0,\ x \geq 0$. Compute the sum of the values of $x^{1/6}$.

(A) 9 (B) 10 (C) 11 (D) 15 (E) 20

14. Anthony's mother is 20 years older than Anthony, but she is 3 years younger than Anthony's father. Anthony's father is 7 years younger than 3 times Anthony's age. Find the sum of their ages.

(A) 66 years (B) 77 years (C) 88 years (D) 102 years (E) None of these

15. Let $f(x)$ be a real function such that $f(x) = x^{17} - x$ where x is a real number. Find the number of real roots of $f(x)$.

(A) 8 (B) 4 (C) 3 (D) 2 (E) 1

16. A rectangular storage container with an open top has a volume of $10\ m^3$. The length of its base is twice its width. Material for the base costs $10 per square meter; material for the sides costs $6 per square meter. Express the cost *C* of material as a function of the width of the base *w*.

(A) $C(w) = 10w^2 + \frac{180}{w}, w > 0$ (B) $C(w) = 20w^2 + \frac{360}{w}, w > 0$

(C) $C(w) = 20w^2 + \frac{180}{w}, w > 0$ (D) $C(w) = 10w^2 + \frac{360}{w}, w > 0$

(E) None of these

17. Compute the maximum value of $P(x, y) = 4x + 5y$ subject to the constraints $x \geq 0,\ y \geq 0,\ 2x + 2y \leq 10$, and $x + 2y \leq 6$.

(A) 19 (B) 20 (C) 21 (D) 22 (E) None of these

18. Consider the inequality $|x - 2| \leq 7$. Let $(a_1,\ a_2)$ be an ordered pair such that $a_1 \leq \frac{1}{x-10} \leq a_2$. Compute $(a_1,\ a_2)$.

(A) $(-15, -1)$ (B) $(-1, -15)$ (C) $\left(-1, -\frac{1}{15}\right)$ (D) $\left(-\frac{1}{15}, -1\right)$ (E) $\left(1, \frac{1}{15}\right)$

19. Determine the last digit of 2018^{2018}.

(A) 2 (B) 4 (C) 6 (D) 8 (E) None of these

20. Let $a = \frac{x}{2018} - 2018,\ b = \frac{x}{2018} - 2016$, and $c = \frac{x}{2018} - 2020$ where $x \neq 0$. Find the value of $a^2 + b^2 + c^2 - ab - bc - ca$.

(A) 4 (B) 8 (C) 12 (D) 16 (E) 24

21. For what value of x, does the equation, $8y+7=2y^2+3x$, have a unique solution for y?

(A) 0 (B) 1 (C) 3 (D) 5 (E) none of these

22. Which of the following relations describes the points that are equidistant from (6, 0) and (0, 8)?

(A) $(x-6)^2+(y-8)^2=100$ (B) $x^2+(y-8)^2=100$ (C) $(x-6)^2+y^2=100$

(D) $4x+3y=24$ (E) $3x-4y=-7$

23. Determine the area of a right triangle whose hypotenuse is 50 cm and whose perimeter is 112 cm.

(A) 336 cm^2 (B) 480 cm^2 (C) 672 cm^2 (D) 1,550 cm^2
(E) 2,800 cm^2

24. Four coins are pulled from a jar that contains four nickels, five dimes and a quarter. Assuming that all the coins have an equal chance of being pulled, what is the probability that the value of the four coins is exactly 40¢?

(A) $\frac{2}{105}$ (B) $\frac{1}{84}$ (C) $\frac{3}{70}$ (D) $\frac{7}{400}$ (E) $\frac{1}{42}$

25. A trapezoid with two internal angles measuring 60º has three sides of length 10 cm. Determine its area.

(A) $25\sqrt{3}\ \text{cm}^2$ (B) $150\ \text{cm}^2$ (C) $10+25\sqrt{3}\ \text{cm}^2$ (D) $75\sqrt{3}\ \text{cm}^2$ (E) none of these

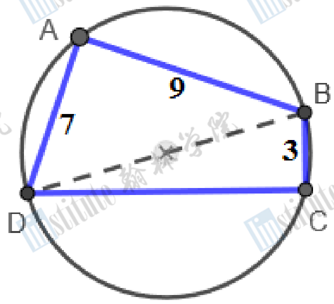
26. A triangle with vertices, (0, 4), (3, 0) and (x, 2x), has area of measure 42. If $x > 0$, determine x.

(A) 8.4 (B) 9.6 (C) $4\sqrt{6}$ (D) 36 (E) none of these.

27. Barbara wants to raise her average in English class to at least a 93 by doing well on her term paper. At the present, she has a 98 on her homework grade, a 95 on her participation grade, and an 89 as her test average. Suppose homework counts as 10%, participation as 20%, the test average as 40%, and the term paper makes up the final part of her grade. What is the minimum grade that she needs on the term paper to achieve her goal? (Do not allow rounding, and assume that the grade on the term paper is a whole number.)

(A) 90 (B) 92 (C) 93 (D) 95 (E) 96

28. A quadrilateral inscribed in a circle has sides with lengths 3, 7, and 9 as shown. The length of *DC*, the fourth side is not known. What is the length *DC*, if *BD* passes through the center of the circle?



(A) 11 (B) $\sqrt{130}$ (C) 12 (D) $\sqrt{139}$ (E) 13

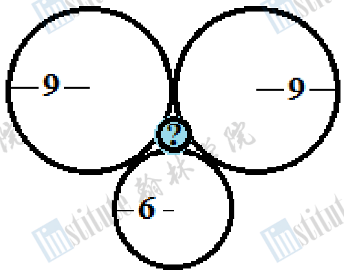
29. How many distinct real solutions does the equation, $x^5 - x^3 + x = x^4 + x^3 - x^2$ have?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

30. A circle centered at (3, 4) intersects a line containing the point (4, 3) at two points, one of which is the origin. How long is the chord that connects the two intersection points?

(A) 9.6 (B) $\sqrt{96}$ (C) 10 (D) $\sqrt{108}$ (E) 14

31. Four mutually tangent circles are arranged as shown. Determine the radius of the middle circle given the outer three have radii of lengths 6, 9, and 9.



(A) 1.2 (B) 1 (C) 1.5 (D) $\sqrt{3}$ (E) 3

32. What is the product of the values *a*, *b*, and *c* that require the graph $y = ax^2 + bx + c$ to pass through the points (0,4), (1,5), and (2,2)?

(A) 40 (B) –24 (C) –6 (D) 12 (E) none of these

33. A 25 foot ladder leans against the side of a house. The foot of the ladder is 20 feet from the house. If the top of the ladder slips down 8 feet, how far will the foot of the ladder move from its original position?

(A) 3 feet (B) 4 feet (C) 5 feet (D) 24 feet (E) none of these

34. The probability that a randomly chosen college student lives on campus is 40%. The probability that a student is registered to vote is 75%. The probability that a college student neither lives on campus nor is registered to vote is 10%. What is the probability that a randomly selected student both lives on campus and is registered to vote?

(A) 75% (B) 45% (C) 30% (D) 25% (E) none of these

35. A circle is inscribed inside a regular hexagon. A second regular hexagon is inscribed inside this circle. Find the ratio of the area of the large hexagon to the area of the small hexagon.

(A) 2:1 (B) 3:2 (C) $\sqrt{3}:1$ (D) $2:\sqrt{3}$ (E) none of these

36. Luigi's Pizza Parlor advertises 84 different three-topping pizzas. Assume he uses all the different combinations of their individual toppings, how many toppings does Luigi actually use?

(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

37. Calculate the sum, $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\Lambda+\frac{1}{420}$.

(A) 1 (B) $\frac{197}{210}$ (C) $\frac{19}{20}$ (D) $\frac{419}{420}$ (E) $\frac{20}{21}$

38. The sum of two real numbers is 21, and the difference of their squares is 63. Find the product of the two numbers.

(A) 98 (B) 101.25 (C) 104 (D) 108 (E) 110

39. A triangle has area equal to $6\sqrt{6}$ square inches. One side is 6 inches long. Another side is 7 inches long. Which of these could be the length of the missing side.

(A) 8 inches (B) $\sqrt{13}$ inches (C) 5 inches (D) $\sqrt{84}$ inches (E) 9 inches

40. Determine the exact value of $\cos(\sin^{-1}(-0.96))$?

(A) 0.28 (B) 0.04 (C) $\tan(-0.96)$ (D) -0.28 (E) 0.96