NC High School Mathematics Contest Finals 2018: Level I

1. Find the number of integers that are in the region defined by the inequality $5 < |n| \leq 73$.

   (A) 136 (B) 134 (C) 68 (D) 67 (E) None of these

2. Let $n, n+1$ be positive intergers such that their quotient is 1.02. Compute the product of n and $n+1$.

   (A) 101 (B) 202 (C) 2550 (D) 5250 (E) None of these

3. Consider the function $f(x) = \frac{x^6+3x^5-4x^4-12x^3}{x^4-2x^3}$ with domain $D = \{x|\ x \neq 0,\ x \neq 2\}$. Evaluate $f(9997)$.

   (A) 9997000 (B) 99997000 (C) 9999900 (D) 99990000 (E) None of these

4. How many real solutions are there to the equation
   $\left|200x + |1000x - 800|\right| = 400$?

   (A) 0 (B) 1 (C) 2 (D) 4 (E) None of these

5. Suppose a point is selected at random inside a circle. Find the probability that the point is closer to the center of the circle than the edge.

   (A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) $\frac{5}{8}$ (E) $\frac{7}{8}$

6. In an isosceles triangle ABC, $m\angle C = 30^0$ and $AC = BC$. Suppose $AC = a$. Find BD if D is on $\overline{BC}$ such that $\overline{AD}$ is the altitude of triangle ABC.

   (A) $\frac{(3-\sqrt{5})a}{2}$ (B) $\frac{(\sqrt{5}-1)a}{2}$ (C) $\frac{(\sqrt{3}-1)a}{2}$ (D) $\frac{(2-\sqrt{3})a}{2}$ (E) $\frac{1}{3}a$

7. Consider the sequence $\{a_n\}$ that is defined recursively by
$a_1 = 10, \quad a_{n+1} = \sqrt{a_n}$ for $n \geq 1$. If the sequence is continued indefinitely, which of the following numbers gives the limit of $\{a_n\}$?
(A) $2\sqrt{5}$ (B) $\sqrt{21}$ (C) 3 (D) 5 (E) 1

8. Three athletes run a 1500 meter- race. When the first runner finishes, the second runner and third runner are at 100 and 170 meters from the finish line, respectively. If each runner maintains a constant speed, how far is the third runner from the finish line when the second runner finishes?

(A) $70\ m$ (B) $72.5\ m$ (C) $74.5\ m$ (D) $75\ m$ (E) $77\ m$

9. Find the area of the regions enclosed by the equations:
$|x| - |y| = 1$ and $|x| = 3$.

(A) $12\ unit^2$ (B) $8\ unit^2$ (C) $6\ unit^2$ (D) $5\ unit^2$ (E) $3\ unit^2$

10. A virulent infectious disease spreads as given by the equation $N(t) = N_0(1 - 10^{-kt})$ where:
t is time in hours,
$N(t)$ is number of people infected at time t
N_0 is the total population of the community at time $t = 0$
k is the infection rate constant of the disease ($k = 0.001$).
Calculate how long will it take for 90% of the population to be infected.

(A) $90\ hours$ (B) $100\ hours$ (C) $1000\ hours$ (D) $990\ hours$ (E) None of these

11. Let m be the solution of the equation

$$\frac{4x+9}{5} - \frac{2x+3}{3} = \frac{x-5}{2}$$

Compute the value of $m^2 + m$.

(A) 61 (B) 72 (C) 80 (D) 85 (E) 90

12. Simplify the expression:

$$\left(\frac{1}{1+\sqrt[3]{2}+\sqrt[3]{4}}\right)^3$$

(A) $1+3\sqrt[3]{2}-3\sqrt[3]{4}$ (B) $1-\sqrt[3]{2}-\sqrt[3]{4}$ (C) $1-3\sqrt[3]{2}+3\sqrt[3]{4}$ (D) $\sqrt[3]{2}-1$
(E) None of these

13. Let p be the solution of the equation $5^{2x-1}\cdot 10^{6x}=8^{2x}$. Find the value of $\left(\sqrt[3]{p}+p\right)^2$.

(A) 2 (B) $\frac{2}{3}$ (C) $\frac{1}{64}$ (D) $\frac{25}{64}$ (E) None of these

14. Consider the sequence $\{a_n\}$ such that $a_0=4,\ a_1=2,\ a_n=a_{n-1}-a_{n-2}$. Compute the sum: $a_0+a_1+a_2+\cdots+a_{101}$.

(A) 4 (B) 2 (C) -2 (D) -4 (E) 0

15. Consider the real function $f(x)=ax^3-bx-5$ such that $f(1)=0$. Find $f(-1)$.

(A) 1 (B) -1 (C) -5 (D) -10 (E) -20

16. Suppose the quadratic equation $x^2+kx+2k=0$ has at least one real solution. Find the criteria satisfied by k.

(A) $k\le -1$ (B) $k\ge 9$ (C) $k\le -1$ or $k\ge 9$ (D) $k\le 0$ or $k\ge 9$ (E) $k\le 0$ or $k\ge 8$

17. Suppose the function $f(x)=x^2+2bx+b$ has two x-intercepts such that one of the x intercepts has a value of -3. Compute the value of the other x-intercept.

(A) $\frac{1}{2}$ (B) -2 (C) $-\frac{5}{3}$ (D) $-\frac{3}{5}$ (E) $\frac{6}{5}$

18. A triangle has sides of lengths *4*, *5*, and *6*. Find the length of the altitude to the side of length *6*.

(A) $\frac{5\sqrt{7}}{3}$ (B) $\frac{15}{2}$ (C) $\frac{5\sqrt{7}}{4}$ (D) $\frac{24}{5}$ (E) $\frac{5\sqrt{7}}{6}$

19. Consider the triangle *ABC* with vertices with coordinates: $A = (4, -3)$, $B = (4, 1)$, and $C = (2,1)$. Calculate the area of the triangle *ABC*.

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) 4 (E) None of these

20. Consider the function $f(x) = x^2$. Suppose the following sequence of transformations are performed on $f(x)$ to obtain $g(x)$.

i. Translation of 2 units to the left.
ii. Vertical shift of *3* units up.
iii. Reflection in the *y* axis.
iv. Reflection in the *x* axis.

Evaluate $g(2)$.

(A) -5 (B) -3 (C) 3 (D) 5 (E) 13

21. Let *x* and *y* be real variables such that $x \geq 0$ and $y \geq 0$. Suppose $f(x, y) = (x^2 + y^2 - 2)^2 + 4(xy + 2)(x^2 + xy + y^2)$. Compute $\sqrt{f(x, y)}$.

(A) $x^2 + y^2$ (B) $(x^2 + y^2 - 2)^{1/2} + 2xy$ (C) $2(xy + 2)^2$ (D) $(x + y)^2 + 2$ (E) $4xy$

22. The altitude perpendicular to the hypotenuse of a right triangle is 12 cm. Express the length of the hypotenuse h as a function of the perimeter *P*.

(A) $h = \frac{P^2}{2P+24}$ (B) $h = \frac{P}{2P+34}$ (C) $h = \frac{P^2+24}{2P}$ (D) $h = \frac{2P+24}{P^2}$ (E) $h = \frac{P}{2P^2+24}$

23. Twenty-four different 4-digit numbers can be arranged by re-arranging the digits of the number 1234. Find the sum of these twenty-four numbers.

(A) 57661 (B) 67542 (C) 76563 (D) 66664 (E) 66660

24. Consider the set $S = \{a_1,\ a_2,\ a_3, \dots\ a_n\ \}$. Let the arithmetic mean of S be denoted by $E(S)$ and the variance of S be given by $V(S)$ such that:

$$E(S) = \frac{1}{n}\sum a_i$$

$$V(S) = \frac{1}{n-1}\sum(a_i - E(S))^2$$

If each element of *S* is multiplied by *k* and then increased by *l*, find the value of the new variance of the set *S*.

(A) $E(S) + kl$ (B) $k^2V(S)$ (C) $k^2V(S) + l$ (D) $l^2V(S) + k$ (E) $(E(S))^2 - k^2$

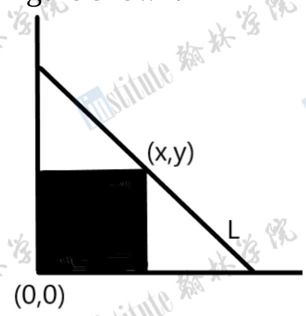
25. Consider the triangle *PQR*. Let the points *A*, *B*, and *C* be the midpoints of the sides *PQ*, *QR*, and *PR*, respectively. Given that the midpoint coordinates are such that $A = (-4, 1)$, $B = (-2, 2)$, and $C = (-5,3)$, compute the area of triangle *PQR*.

(A) 40 $unit^2$ (B) 20 $unit^2$ (C) 10 $unit^2$ (D) 4 $unit^2$ (E) None of these

26. In the mythical city of Jamais, the 6-faced die used in *Casino Mathematicale* has the face numbers $\{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$ handcrafted such that the probability of a number *x* appearing during a random toss is proportional to the number *x*. Compute the probability that when a single die is tossed randomly, an odd prime number is observed.

(A) $\frac{2}{9}$ (B) $\frac{1}{7}$ (C) $\frac{8}{21}$ (D) $\frac{3}{4}$ (E) None of these

27. Consider the shaded rectangle in the figure shown.



If the equation of *L* is $2x + y = 100$, find the value of the maximum possible area of the shaded rectangle.

(A) 1150 $unit^2$ (B) 1250 $unit^2$ (C) 1300 $unit^2$ (D) 1350 $unit^2$ (E) None of these

28. Consider the binary operations © and ® defined over the real numbers by the expressions $a©b = 2^{a+b}$ and $a®b = 2^{ab}$. Suppose $2®(2©2) = 2x©4$. Solve for x.

(A) 4 (B) 6 (C) 12 (D) 14 (E) 32

29. Consider the ordered triple $\{a_1,\ a_2,\ a_3\}$ such that $\frac{1}{1+\sqrt{2}+\sqrt{3}} = a_1 + a_2\sqrt{2} + a_3\sqrt{6}$. Find the sum of a_1, a_2, and a_3.

(A) 0 (B) $\frac{1}{6}$ (C) $\frac{1}{3}$ (D) $\frac{3}{8}$ (E) $\frac{1}{2}$

30. Suppose a, b, c, and d are non-zero real numbers. If $ax^2 + bx + c = 0$ has two real solutions of opposite signs and $ax^2 + bx + c = d$ has two real solutions of the same sign, which of the following statements is true?

(A) $ac < 0,\ |c| < |d|$ (B) $ac < 0,\ |c| > |d|$ (C) $ac > 0,\ |c| < |d|$
(D) $ac > 0,\ |c| > |d|$ (E) $ac > 0,\ c > d$

31. Let x be a real number. Consider the equation $x^{1/2} - 9x^{1/3} + 20x^{1/6} = 0,\ x \geq 0$. Compute the sum of the values of $x^{1/6}$.

(A) 9 (B) 10 (C) 11 (D) 15 (E) 20

32. Anthony's mother is 20 years older than Anthony, but she is 3 years younger than Anthony's father. Anthony's father is 7 years younger than 3 times Anthony's age. Find the sum of their ages.

(A) 66 years (B) 77 years (C) 88 years (D) 102 years (E) None of these

33. Let $f(x)$ be a real function such that $f(x) = x^{17} - x$ where x is a real number. Find the number of real roots of $f(x)$.

(A) 8 (B)4 (C) 3 (D) 2 (E) 1

34. A rectangular storage container with an open top has a volume of 10 m^3. The length of its base is twice its width. Material for the base costs $10 per square meter; material for the sides costs $6 per square meter. Express the cost C of material as a function of the width of the base w.

(A) $C(w) = 10w^2 + \frac{180}{w}, w > 0$ (B) $C(w) = 20w^2 + \frac{360}{w}, w > 0$
(C) $C(w) = 20w^2 + \frac{180}{w}, w > 0$ (D) $C(w) = 10w^2 + \frac{360}{w}, w > 0$ (E) None of these

35. Compute the maximum value of $P(x, y) = 4x + 5y$ subject to the constraints $x \geq 0,\ y \geq 0$, $2x + 2y \leq 10$, and $x + 2y \leq 6$.

(A) 19 (B) 20 (C) 21 (D) 22 (E) None of these

36. Consider the inequality $|x - 2| \leq 7$. Let $(a_1,\ a_2)$ be an ordered pair such that $a_1 \leq \frac{1}{x-10} \leq a_2$. Compute $(a_1,\ a_2)$.

(A) $(-15, -1)$ (B) $(-1, -15)$ (C) $\left(-1, -\frac{1}{15}\right)$ (D) $\left(-\frac{1}{15}, -1\right)$ (E) $\left(1, \frac{1}{15}\right)$

37. Determine the last digit of 2018^{2018}.

(A) 2 (B) 4 (C) 6 (D) 8 (E) None of these

38. Let $a = \frac{x}{2018} - 2018,\ b = \frac{x}{2018} - 2016$, and $c = \frac{x}{2018} - 2020$ where $x \neq 0$. Find the value of $a^2 + b^2 + c^2 - ab - bc - ca$.

(A) 4 (B) 8 (C) 12 (D) 16 (E) 24

39. The probability of rolling two dice and observing a double-1 (making a sum of two, or "snake-eyes"), is 1 in 36. Calculate the probability of getting a double-1 on *at least one of the two rolls of* a pair of unbiased dice.

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{71}{1296}$ (D) $\frac{1}{1296}$ (E) $\frac{1}{18}$

40. Let $f(x) = 1 + x + x^2 + x^3 + x^4$. Find the remainder when $f(x)$ is divided by $x - \frac{1}{10}$.

(A) 10.9999 (B) 9.9999 (C) 3.3333 (D) 2.2222 (E) 1.1111