The Fortieth Annual
State High School
Mathematics Contest

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of Teachers of Mathematics

NC STATE MATHEMATICS CONTEST
APRIL 2018

PART I: 20 MULTIPLE CHOICE PROBLEMS

1. The marked price of a phone was 20% less than the suggested retail price. Andrea purchased the phone for 70% of the marked price at a special, one-day sale. What percent of the suggested retail price did Andrea pay?

   (A) 44% (B) 50% **(C)** 56% (D) 86% (E) 90%

   Solution: Answer: (C). Let x be the retail price of the phone. The marked price was $0.8x$, and Andrea paid 70% of the marked price, which is $0.56x$. Thus, Andrea paid 56% of the retailed price.

2. If $i^2 = -1$, find the sum

$$(-i)^1 + (-i)^2 + (-i)^3 + \cdots + (-i)^{2017} + (-i)^{2018}.$$

   (A) 0 (B) $-i$ (C) $1 - i$ **(D)** $-1 - i$ (E) None of the answers (A) through (D) is correct.

   Solution: Answer: (D). As $(-i)^2 = -1, (-i)^3 = i, (-i)^4 = 1$, we get $(-i)+(-i)^2+(-i)^3+(-i)^4 = 0$. Hence $(-i)^1+(-i)^2+(-i)^3+\cdots+(-i)^{2016}+(-i)^{2017}+(-i)^{2018} = (-i)^{2017}+(-i)^{2018} = (-i)+(-i)^2 = -1 - i$.

3. If a and b are real numbers and z is a solution of the equation $z^2 + z + 1 = 0$, then $(az^2 + bz)(bz^2 + az)$ is equal to

   (A) $a^2 + ab + b^2$ **(B)** $a^2 - ab + b^2$ (C) $a^2 + a + b^2$ (D) $a^2 - b + b^2$
   (E) None of the answers (A) through (D) is correct.

   Solution: Answer: (B). Multiplying $z^2 + z + 1 = 0$ by $z - 1$, we get $z^3 = 1$, which implies $z^4 = z$. Then

$$(az^2+bz)(bz^2+az) = abz^4+a^2z^3+b^2z^3+abz^2 = abz+a^2+b^2+abz^2 = a^2+b^2+ab(z^2+z) = a^2+b^2-ab.$$

4. Let a and b be real numbers and $a \neq 0$. Assume that the equations $ax^2+bx+b = 0$ and $ax^2+ax+b = 0$ have real roots. If the product of one root of the equation $ax^2 + bx + b = 0$ and one root of the equation $ax^2 + ax + b = 0$ is 1, what is $a + b$?

   (A) -2 (B) -1 (C) 2 (D) 3
   (E) None of the answers (A) through (D) is correct.

   Solution: Answer: (E). Let y be a solution of $ax^2+bx+b = 0$ and z be a solution of $ax^2+ax+b = 0$ such that $yz = 1$. Then $z = \frac{1}{y}$ and $a\frac{1}{y^2} + a\frac{1}{y} + b = 0$, i.e. $by^2 + ay + a = 0$. Since $ay^2 + ay + b = 0$, we get $(a+b)y^2+(a+b)y+(a+b) = 0$, i.e $(a+b)(y^2+y+1) = 0$. Since $y^2+y+1 \neq 0$, we conclude $a + b = 0$.

5. Find the number of all ordered pairs of positive integers (x, y) such that $20x + 18y = 2018$.

(A) 8 (B) 9 (C) 10 (D) 11 **(E) 12**

> **Solution:** Answer: (E). From the given equation we get $10x + 9y = 1009$. Then $9y = 1009 - 10x = 10(100-x)+9 = 9(100-x)+9+(100-x)$. From the last equation we obtain $y = 100-x+1+\frac{1}{9}(100-x)$. Hence, $9|(100-x)$, i.e. $x \in \{1, 10, 19, 28, 37, 46, 55, 64, 73, 82, 91, 100\}$. All these values of x yield a positive integer value for y. Therefore, the equation has 12 positive integer solutions.

6. Let $p(x) = x^4 + ax^3 + bx^2 + cx + d$ be a polynomial with real coefficients. If $1 + i$ and i are roots of the polynomial $p(x)$, find $a + b + c + d$.

(A) -1 (B) 0 **(C)** 1 (D) 2 (E) 3

> **Solution:** Answer: (C). Since $1 + i$ and i are roots of the polynomial $p(x)$ whose coefficients are real, it follows that their conjugates $1 - i$ and $-i$ are also roots of $p(x)$. Then
>
> $$p(x) = (x-1-i)(x-1+i)(x-i)(x+i) = (x^2-2x+2)(x^2+1) = x^4 - 2x^3 + 3x^2 - 2x + 2.$$
>
> Thus, $a + b + c + d = 1$.

7. Let M be a point on the side $\overline{BC}$ in the triangle ABC. The line passing through M and parallel to $\overline{AB}$ intersects the side $\overline{AC}$ at a point L. The line passing through M and parallel to $\overline{AC}$ intersects the side $\overline{AB}$ at a point K. The areas of the triangles KBM and LMC are 4 and 9, respectively. Find the area of the quadrilateral $AKML$.

(A) 5 (B) 6 (C) 10 **(D)** 12 (E) 13

> **Solution:**
>
> Answer: (D). Let a be the area of $\triangle ABC$. Since $\triangle KBM$ and $\triangle LMC$ are similar to $\triangle ABC$, we get
>
> $$\sqrt{\frac{4}{a}} = \frac{BM}{BC}, \quad \sqrt{\frac{9}{a}} = \frac{CM}{BC}.$$
>
> Then $\sqrt{\frac{4}{a}} + \sqrt{\frac{9}{a}} = \frac{BM+CM}{BC} = 1$. Thus, $\sqrt{a} = \sqrt{4} + \sqrt{9} = 5$, i.e. $a = 25$. Therefore the area of the quadrilateral $AKML$ is $25 - 4 - 9 = 12$.
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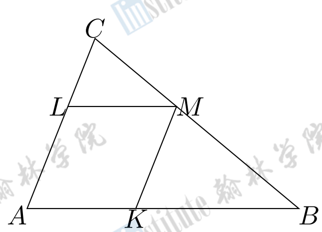
> 
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8. In the set $A = \{3, 6, 9, 10, n\}$ the element n is an integer not equal to any of the other four elements of the set A. If the median of the elements of the set A is equal to the mean of the elements of the set A, find the sum of all possible values of n.

(A) 9 (B) 19 (C) 24 **(D)** 26 (E) None of the answers (A) through (D) is correct.

Solution: Answer: (D). The mean of the set A is $\frac{28+n}{5}$. If $n < 6$, then the median of A is 6, so $\frac{28+n}{5} = 6$, which implies $n = 2$. If $6 < n < 9$, then the median of A is n, so $\frac{28+n}{5} = n$, which implies $n = 7$. If $n > 9$, then the median of A is 9, and $\frac{28+n}{5} = 9$ implies $n = 17$. Therefore, the sum of all possible values of n is 26.

9. What is the product of the real solutions of the equation $(x + 1)(x + 2)(x + 3)(x + 4) = 3$?
(A) -3 (B) 3 (C) 10 (D) 21 (E) None of the answers (A) through (D) is correct.

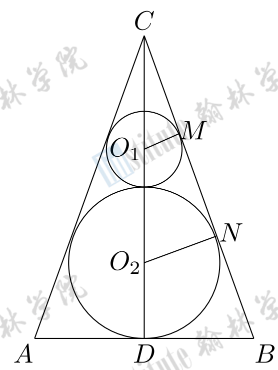
Solution: Answer: (B). Notice $(x + 1)(x + 4) = x^2 + 5x + 4$ and $(x + 2)(x + 3) = x^2 + 5x + 6$. If we introduce a substitution $t = x^2 + 5x + 4$, then the given equation is equivalent to $t(t + 2) = 3$, whose solutions are $t = -3$ and $t = 1$. The equations $x^2 + 5x + 4 = -3$ has complex solutions and the solutions of the equation $x^2 + 5x + 4 = 1$ are $-\frac{5}{2} \pm \frac{\sqrt{13}}{2}$. Thus, the product of the real solutions is 3.

10. What is the largest number of acute angles that a convex heptagon (7-sided polygon) can have?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Solution: Answer: (D). The sum of the angles in a convex heptagon is 900° and each angle is less than 180°. If four of the angles are acute, then their sum would be less than 360°, which would imply that the sum of the other three angles is at lest 540°. Hence, at least one of the non-acute angles is at least 180°, a contradiction. Thus, it can be at most three acute angles in a convex heptagon. Since there is a convex heptagon with exactly three acute angles (draw such a heptagon), the largest number of acute angles is 3.

11. Two spheres with radii 1 and 2 are inscribed in a cone. The larger sphere touches the base and the lateral surface of the cone, and the smaller sphere touches the lateral surface of the cone and the larger sphere. Find the volume of the cone.
(A) $\frac{64\pi}{3}$ (B) $\frac{48\sqrt{3}\pi}{3}$ (C) $18\sqrt{3}\pi$ (D) 24π (E) None of the answers (A) through (D) is correct.

Solution: Answer: (A). Since the triangles CO_1M and CO_2N are similar, we get $CO_1 : CO_2 = O_1M : O_2N$, i.e. $CO_1 : (CO_1 + 3) = 1 : 2$. Thus, $CO_1 = 3$. The height of the cone is $h = 8$. By Pythagorean Theorem, $CM = \sqrt{O_1C^2 - O_1M^2} = 2\sqrt{2}$. Since the triangles CMO_1 and CDB are similar, we get $O_1M : CM = DB : CD$. Hence, $DB = 2\sqrt{2}$. The volume of the cone is $V = \frac{1}{3}DB^2\pi h = \frac{64\pi}{3}$.



12. Determine the sum of all values of the integer a such that the roots of the equation $x^2-2ax-(a+3)=0$ are integers.

(A) -2 **(B)** -1 (C) 1 (D) 3 (E) 4

Solution: Answer: (B). The solutions of the given equation are $a \pm \sqrt{a^2+a+3}$. Both solutions are integers if $a^2+a+3=k^2$ for some integer k. If we solve the last equation for a we get $a=\frac{-1\pm\sqrt{4k^2-11}}{2}$. Hence, a is an integer if $4k^2-11=n^2$ for some integer n. From the last equation we get $(2k-n)(2k+n)=11$. Then we get the following four systems: $2k-n=1, 2k+n=11$; $2k-n=-1, 2k+n=-11$; $2k-n=11, 2k+n=1$; $2k-n=-11, 2k+n=-1$. We get that $k=3$ or $k=-3$, which imply $a=-3$ or $a=2$.

13. The sides of a triangle have length 11, 15, and x, where x is an integer. For how many values of x is the triangle obtuse?

(A) 6 (B) 7 (C) 12 **(D)** 13 (E) 17

Solution: Answer: (D). Using the triangle inequality, we get $4<x<26$. The triangle is obtuse if either $11^2+15^2<x^2$ or $11^2+x^2<15^2$. There are 13 integers for x that satisfy one of the previous inequalities: 5,6,7,8,9,10,19,20,21,22,23,24, and 15.

14. Let a and b be the number of digits in 2^{2018} and 5^{2018} respectively. Find $a+b$.

(A) 2017 (B) 2018 **(C) 2019** (D) 2020 (E) 2021

Solution: Answer: (C). Since $10^{a-1}<2^{2018}<10^a$ and $10^{b-1}<5^{2018}<10^b$, we have $10^{a-1}\cdot 10^{b-1}<2^{2018}\cdot 5^{2018}<10^a\cdot 10^b$, i.e. $10^{a+b-2}<10^{2018}<10^{a+b}$. Thus $a+b-1=2018$, which implies $a+b=2019$.

15. A closed right circular cylinder has an integer radius and an integer height. The numerical value of its volume is five times the numerical value of its surface area. Determine how many distinct cylinders satisfy this property.

(A) 5 (B) 6 (C) 8 **(D) 9** (E) 10

Solution: Answer: (D). Let r and h be the radius and the height of the cylinder. Then $V=\pi r^2 h$ and $A=2\pi r^2+2\pi rh$. From $\pi r^2 h=5(2\pi r^2+2\pi rh)$ we get $rh=10r+10h$, which is equivalent to $(r-10)(h-10)=100$. Since $r\geq 1$ and $h\geq 1$, both $r-10$ and $h-10$ must be positive divisors of 100. The number of positive divisors of $100=2^2\cdot 5^2$ is $(2+1)(2+1)=9$. Therefore, there are 9 closed right cylinders that satisfy the requirements of the question.

16. If a number is selected at random from the set of five-digit numbers in which the sum of the digits is equal to 43, what is the probability that this number is divisible by 11?

(A) $\frac{2}{5}$ **(B)** $\frac{1}{5}$ (C) $\frac{1}{6}$ (D) $\frac{1}{11}$ (E) None of the answers (A) through (D) is correct.

Solution: Answer: (B). Let $\overline{abcde}$ be a five-digit number. The sum of its digits is at most 45. Since the number that is selected at random has digits whose sum is 43, we conclude that four of the digits are 9 and one digit is 7, or three digits are 9 and two digits are 8. If one of the digits is 7, the following five numbers have sum of their digits 43: 79999, 97999, 99799, 99979, and 99997. The following 10 numbers have three digits 9 and two digits 8: 88999, 89899, 89989, 89998, 98899, 98989, 98998, 99889, 99898, 99988. The five-digit number $\overline{abcde}$ is divisible by 11 if and only if the number $a-b+c-d+e$ is divisible by 11. From the 15 five-digit numbers whose sum of their digits is 43, the following are divisible by 11: 97999, 99979, and 98989. Therefore, the probability we are looking for is $\frac{3}{15} = \frac{1}{5}$.

17. How many real solutions are there for the equation

$$2^{-|x|} = \frac{1}{2\sqrt{2}}(|x+1| - |x-1|)?$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Solution: Answer: (B). Since

$$|x+1| = \begin{cases} x+1, & \text{if } x \geq -1 \\ -(x+1), & \text{if } x < -1 \end{cases}$$

and

$$|x-1| = \begin{cases} x-1, & \text{if } x > 1 \\ -(x-1), & \text{if } x \leq 1 \end{cases}$$

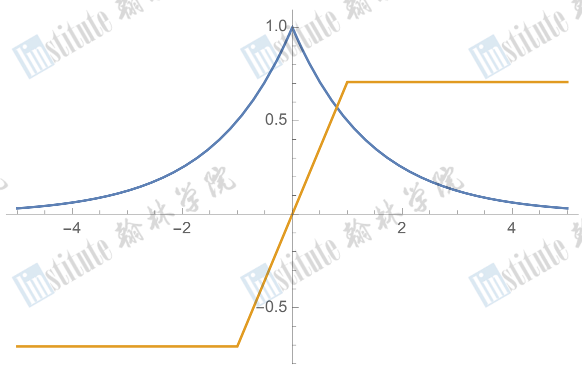
we get

$$g(x) = \frac{1}{2\sqrt{2}}(|x+1| - |x-1|) = \begin{cases} -\frac{1}{\sqrt{2}}, & \text{if } x < -1 \\ \frac{x}{\sqrt{2}}, & \text{if } -1 \leq x \leq 1 \\ \frac{1}{\sqrt{2}}, & \text{if } x > 1 \end{cases}$$

On the other hand

$$f(x) = 2^{-|x|} = \begin{cases} 2^{-x}, & \text{if } x \geq 0 \\ 2^{x}, & \text{if } x < 0 \end{cases}$$

Sketching the graphs of $f(x)$ and $g(x)$ we see that the given equation has one real solution.



18. Find $\sum_{n=1}^{2018} a_n$ if $a_1, a_2, a_3, \ldots$ is a sequence given by

$$a_1 = \frac{1}{2},\ a_n = \frac{a_{n-1}}{2na_{n-1}+1} \text{ for all integers } n > 1.$$

(A) $\frac{2017}{2018}$ (B) $\frac{2018}{2017}$ **(C)** $\frac{2018}{2019}$ (D) $\frac{2019}{2018}$ (E) None of the answers (A) through (D) is correct.

Solution: Answer: (C). We can re-write a_n as

$$a_n = \frac{a_{n-1}}{2na_{n-1}+1} = \frac{1}{2n + \frac{1}{a_{n-1}}}.$$

Then $\frac{1}{a_n} = 2n + \frac{1}{a_{n-1}}$. Thus, $\frac{1}{a_2} - \frac{1}{a_1} = 4$, $\frac{1}{a_3} - \frac{1}{a_2} = 6$, $\frac{1}{a_4} - \frac{1}{a_3} = 8$, $\cdots$, $\frac{1}{a_n} - \frac{1}{a_{n-1}} = 2n$. If we add the previous equations, we get $\frac{1}{a_n} - \frac{1}{a_1} = 4 + 6 + 8 + \cdots + 2n$. Since $a_1 = \frac{1}{2}$, we get $\frac{1}{a_n} = 2 + 4 + 6 + \cdots + 2n = n(n+1)$. Thus, $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Therefore,

$$\sum_{n=1}^{2018} a_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{2018} - \frac{1}{2019} = 1 - \frac{1}{2019} = \frac{2018}{2019}.$$

19. Let a and b ($a > b$) be the legs of a right triangle ABC such that $\log \frac{a-b}{\sqrt{2}} = \frac{1}{2}(\log a + \log b)$. Find the absolute value of the difference of the acute angles, in degrees, of $\triangle ABC$.

(A) $18°$ (B) $30°$ (C) $36°$ (D) $48°$ **(E)** $60°$

Solution: Answer: (E). From $\log \frac{a-b}{\sqrt{2}} = \frac{1}{2}(\log a + \log b)$ we get $\frac{a-b}{\sqrt{2}} = \sqrt{ab}$, i.e. $a - b = \sqrt{2ab}$. If we square both sides of the last equation we get $a^2 + b^2 - 4ab = 0$. Since a and b are nonzero, we have $\left(\frac{a}{b}\right)^2 - 4\left(\frac{a}{b}\right) + 1 = 0$. Let α be the angle opposite the leg a. Then $\tan\alpha = \frac{a}{b}$, and $(\tan\alpha)^2 - 4\tan\alpha + 1 = 0$. If we solve the last equation for $\tan\alpha$, we get $\tan\alpha = 2 - \sqrt{3}$ or $\tan\alpha = 2 + \sqrt{3}$. Since $\tan 15° = \tan(45° - 30°) = \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°} = 2 - \sqrt{3}$, and similarly, $\tan 75° = 2 + \sqrt{3}$, we get that $\alpha = 15°$ or $\alpha = 75°$. Since $a > b$, then $\alpha = 75°$. The other acute angle is $15°$ and the absolute value of their difference is $60°$

20. Determine the minimum value of the function

$$f(x, y, z) = \frac{x^2 + y^2 + z^2}{xy + yz},\ x > 0, y > 0, z > 0.$$

(A) 1 (B) $\frac{3}{2}$ **(C)** $\sqrt{2}$ (D) $\sqrt{3}$ (E) None of the answers (A) through (D) is correct.

Solution: Answer: (C). Using the inequality between Arithmetic Mean and Geometric Mean we get:

$$x^2 + \frac{1}{2}y^2 \geq 2\sqrt{\frac{1}{2}x^2y^2} = \sqrt{2}xy,$$

$$\frac{1}{2}y^2 + z^2 \geq 2\sqrt{\frac{1}{2}y^2z^2} = \sqrt{2}yz.$$

Then $f(x, y, z) = \frac{x^2+y^2+z^2}{xy+yz} \geq \frac{\sqrt{2}(xy+yz)}{xy+yz} = \sqrt{2}$. Since $f(1, \sqrt{2}, 1) = \sqrt{2}$, we get that the minimum of the function is $\sqrt{2}$.

PART II: 10 INTEGER ANSWER PROBLEMS

1. Find $(\cot 1^\circ - \tan 1^\circ)\tan 2^\circ$.

Solution: Answer: 2. $(\cot 1^\circ - \tan 1^\circ)\tan 2^\circ = \left(\frac{\cos 1^\circ}{\sin 1^\circ} - \frac{\sin 1^\circ}{\cos 1^\circ}\right)\tan 2^\circ = \frac{\cos^2 1^\circ - \sin^2 1^\circ}{\sin 1^\circ \cos 1^\circ}\tan 2^\circ = \frac{2\cos 2^\circ}{\sin 2^\circ}\tan 2^\circ = 2\cot 2^\circ \tan 2^\circ = 2$.

2. Find the number of positive integers n such that $n+2$ divides n^5+2.

Solution: Answer: 6. Let n be a positive integer such that $n+2$ divides n^5+2. Since $n^5+2 = n^5+32-30 = (n^5+2^5)-30 = (n+2)(n^4-2n^3+4n^2-8n+16)-30$, it follows that $n+2$ divides 30. Thus, $n+2 \in \{1,2,3,5,6,10,15,30\}$. Since n is a positive integer, $n \in \{1,3,4,8,13,28\}$.

3. Determine the number of all real numbers x for which the numbers $\sqrt{x^2+2x+1}$, $\frac{x^2+3x-1}{3}$, $x-1$, in the given order, are consecutive members of an arithmetic progression.

Solution: Answer: 3. Since $\sqrt{x^2+2x+1}$, $\frac{x^2+3x-1}{3}$, $x-1$, in the given order, are consecutive members of an arithmetic progression, we have $2\frac{x^2+3x-1}{3} = \sqrt{x^2+2x+1}+x-1$, i.e. $2(x^2+3x-1) = 3(\sqrt{(x+1)^2}+x-1)$. Since $\sqrt{(x+1)^2} = |x+1|$, we get $2(x^2+3x-1) = 3(|x+1|+x-1)$, i.e. $2x^2+3x-3|x+1|+1=0$. If $x \geq -1$, then $|x+1| = x+1$, which implies $2x^2-2=0$. The solutions of the last equations are $x=-1$ and $x=1$. If $x<-1$, then $|x+1| = -(x+1)$; in this case we have $2x^2+6x+4=0$ and the solution in this case is $x=-2$ (note that the second solution of this quadratic equation does not satisfy $x<-1$). Therefore, there three real numbers, $-2,-1,1$, for which $\sqrt{x^2+2x+1}$, $\frac{x^2+3x-1}{3}$, $x-1$ are consecutive members of an arithmetic progression.

4. Three times Andrew's age plus Bekir's age equals twice Claudio's age. Double the cube of Claudio's age is equal to the cube of Bekir's age added to three times the cube of Andrew's age. Their respective ages are relatively prime to each other. How old is Bekir?

Solution: Answer: 5. Denote Andrew's, Bekir's, and Claudio's ages by A, B, C, respectively. Then $3A+B=2C$ and $2C^3=3A^3+B^3$. The previous two equations can be re-written as $2(C-A)=A+B$ and $2(C^3-A^3)=A^3+B^3$. The last equation is equivalent to $2(C-A)(C^2+CA+A^2) = (A+B)(A^2-AB+B^2)$. Since $C-A \neq 0$ and $A+B \neq 0$, and $2(C-A)=A+B$, we get $C^2+CA+A^2 = A^2-AB+B^2$. The last equation is equivalent to $B^2-C^2 = A(B+C)$, i.e. $(B+C)(B-C) = A(B+C)$. Thus, $A=B-C$. From the last equation and $3A+B=2C$ we get $C=4A$. Since A and C are relatively prime, $A=1$ and $C=4$. Then $B=2C-3A=5$.

5. Determine the positive integer n such that each of the digits $0,1,2,\ldots,9$ shows up exactly once in either n^3 or n^4, but not in both, i.e. if a digit is used in n^3, then it is not used in n^4 and vice versa.

Solution: Answer: 18. If n is a one digit positive integer, then $n^3 \le 9^3 = 729$ and $n^4 \le 9^4 = 6461$. Thus, at most seven digits are used in n^3 and n^4. If n is a three-digit positive integer, then $n^3 \ge 10^6$ and $n^4 \ge 10^8$. Thus, at least 16 digits are used in n^3 and n^4 which forces some of the digits to repeat. Therefore, n is a two-digit positive integer. If n^3 has five digits, then $n^3 \ge 10^4$, which implies $n \ge 22$. If $n \ge 22$, then n^4 has at least six digits. Hence, n^3 has four digits, and n^4 has six digits. This implies $1000 \le n^3 \le 9999$ and $100000 \le n^4 \le 999999$. Thus, $10 \le n \le 21$ and $18 \le n \le 31$. If $n = 21$, then both n^3 and n^4 end in 1; if $n = 20$, then both n^3 and n^4 end in 0. If $n = 19$, then $19^4 = 130321$ has repeated digits. Hence $n = 18$. We check that $18^3 = 5832$ and $18^4 = 104976$.

6. Find $\sum_{n=1}^{2018} a_n$, if $a_1, a_2, a_3, \ldots$ is a sequence of integers that satisfy

$$1 + \sum_{d|n} (-1)^{\frac{n}{d}} a_d = 0, \quad \text{for } n = 1, 2, 3 \ldots .$$

Solution: Answer: 2047. We find the first 10 terms of the sequence: $a_1 = 1$, $a_2 = 2$, $a_3 = 0$, $a_4 = 4$, $a_5 = a_6 = a_7 = 0$, $a_8 = 8$, $a_9 = a_{10} = 0$. Using induction, we will prove

$$a_n = \begin{cases} n, & \text{if } n = 2^k \text{ for some nonnegative integer } k \\ 0, & \text{otherwise} \end{cases}$$

Assume that a_i satisfies the above statement for all positive integers i less than or equal to some positive integer k. We now consider a_{k+1}. We can write $k+1$ in the form $k+1 = 2^r s$, where s is an odd integer.

If $s = 1$, then $k+1 = 2^r$. Using the inductive hypothesis we have

$$1 + (1 + 2 + 4 + \cdots + 2^{r-1}) - a_{k+1} = 0,$$

and we get $a_{k+1} = 2^r = k+1$.

If $s > 1$, then

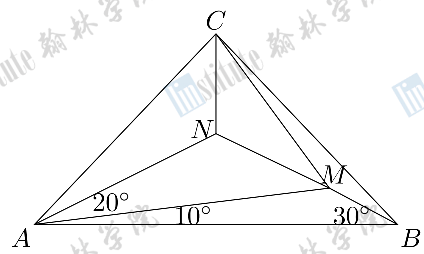
$$1 + (1 + 2 + 4 + \cdots + 2^{r-1}) - 2^r - a_{k+1} = 0,$$

and we get $a_{k+1} = 2^r - 2^r = 0$. Therefore,

$$\sum_{n=1}^{2018} a_n = \sum_{r=0}^{10} a_{2^r} = 1 + 2 + \cdots + 2^{10} = 2^{11} - 1 = 2047.$$

7. Let ABC be a triangle such that $AC = BC$ and $\angle ACB = 80°$. Let M be a point inside $\triangle ABC$ such that $\angle MBA = 30°$ and $\angle MAB = 10°$. In degrees, find $\angle AMC$.

Solution: Answer: 70. Let N be the intersection of the height through C of $\triangle ABC$ and the line passing through B and M. Then $\triangle ABN$ is isosceles (AN=BN) and $\angle ANB = 120°$. Then $\angle BNC = \angle ANC = 120°$. We also get $\angle AMN = \angle MAB + \angle ABM = 10° + 30° = 40°$. Since $\angle ACN = \angle AMN = 40°$, $\angle ANC = \angle ANM$, we get $\angle CAN = \angle NAM$. The triangles ANC and ANM have congruent angles and $\overline{AN}$ is a common side; hence, they are congruent. Thus, $AC = AM$ and $\triangle ACM$ is isosceles with $\angle CAM = 40°$. Hence $\angle AMC = \frac{1}{2}(180° - \angle CAM) = 70°$.



8. Find the positive integer n for which $\frac{20^n+18^n}{n!}$ has maximum value.

Solution: Answer: 19. Let $a_n = \frac{20^n+18^n}{n!}$. Then

$$a_n - a_{n+1} = \frac{1}{n!}(20^n + 18^n) - \frac{1}{(n+1)!}(20^{n+1} + 18^{n+1}) = \frac{1}{(n+1)!}(20^n(n-19) + 18^n(n-17)).$$

Clearly, $a_n - a_{n+1} > 0$ for $n \geq 19$, $a_n - a_{n+1} < 0$ for $n \leq 17$, and $a_{18} - a_{19} = \frac{1}{(n+1)!}(-20^n + 18^n) < 0$. Hence, $a_1 < a_2 < \cdots < a_{18} < a_{19}$ and $a_{19} > a_{20} > \cdots$. Therefore, for $n = 19$, $\frac{20^n+18^n}{n!}$ has maximum value.

9. Determine the number of real solutions of the equation

$$x^2 - \lfloor x^2 \rfloor = (x - \lfloor x \rfloor)^2$$

that are in the interval $[1, 100]$. ($\lfloor a \rfloor$ denotes the largest integer not exceeding a.)

Solution: Answer: 9901. Let $x = n + r$, where $n \in \{1, 2, \ldots, 99\}$ and $0 \leq r < 1$. Then we can re-write the given equation as $n^2 + 2nr = \lfloor n^2 + 2nr + r^2 \rfloor$. For a fixed n, $x = n + r$ is a solution of the last equation if and only of $2nr$ is an integer. Thus, $r \in \left\{0, \frac{1}{2n}, \frac{2}{2n}, \ldots, \frac{2n-1}{2n}\right\}$, and the number of solutions in the interval $[n, n+1)$ is $2n$. It is clear that $x = 100$ is a solution. Therefore, the number of solutions of the given equation that are in the interval $[0, 100]$ is

$$1 + 2(1 + 2 + \cdots + 99) = 1 + 2\frac{99 \cdot 100}{2} = 9901.$$

10. Let a, b, c, d, e, and f be real numbers whose sum is 10 and

$$(a-1)^2 + (b-1)^2 + (c-1)^2 + (d-1)^2 + (e-1)^2 + (f-1)^2 = 6.$$

If the maximum value of f is $\frac{p}{q}$, where p and q are relatively prime positive integers, find $p + q$.

Solution: Answer: 13. From the given conditions we get $a + b + c + d + e = 10 - f$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 20 - f^2$. Using the inequality between Arithmetic Mean and Quadratic Mean, we get $\frac{a+b+c+d+e}{5} \leq \sqrt{\frac{a^2+b^2+c^2+d^2+e^2}{5}}$. Hence, $\frac{(10-f)^2}{5} \leq 20 - f^2$. The last inequality is equivalent to $2f(3f - 10) \leq 0$. Thus, $f_{\max} = \frac{10}{3}$, and it is achieved for $a = b = c = d = e = \frac{4}{3}$.

The following problem will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test.

TIE BREAKER PROBLEM

Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined by

$$f(n) = \begin{cases} n - 10, & \text{if } n > 100 \\ f(f(n+11)), & \text{if } n \leq 100 \end{cases}$$

Find $f(50)$.

Solution: Answer: 91. $f(100) = f(f(100+11)) = f(f(111)) = f(101) = 101 - 10 = 91$. Assume that $f(x) = 91$ for every $x \in \{k+1, k+2, \ldots, 100\}$, where k is an integer. If $90 < k < 100$, then $f(k) = f(f(k+11)) = f(k+1) = 91$. If $k \leq 90$, then $f(k) = f(f(k+1)) = f(91) = 91$. By the backwards Mathematical Induction, we get that for all $k \leq 100$, $f(k) = 91$.