The Fortieth Annual State High School Mathematics Contest

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NC STATE MATHEMATICS CONTEST
APRIL 2018

PART I: 20 MULTIPLE CHOICE PROBLEMS

1. The marked price of a phone was 20% less than the suggested retail price. Andrea purchased the phone for 70% of the marked price at a special, one-day sale. What percent of the suggested retail price did Andrea pay?

   (A) 44% (B) 50% (C) 56% (D) 86% (E) 90%

2. If $i^2 = -1$, find the sum

   $$(-i)^1 + (-i)^2 + (-i)^3 + \cdots + (-i)^{2017} + (-i)^{2018}.$$

   (A) 0 (B) $-i$ (C) $1 - i$ (D) $-1 - i$ (E) None of the answers (A) through (D) is correct.

3. If a and b are real numbers and z is a solution of the equation $z^2 + z + 1 = 0$, then $(az^2 + bz)(bz^2 + az)$ is equal to

   (A) $a^2 + ab + b^2$ (B) $a^2 - ab + b^2$ (C) $a^2 + a + b^2$ (D) $a^2 - b + b^2$
   (E) None of the answers (A) through (D) is correct.

4. Let a and b be real numbers and $a \neq 0$. Assume that the equations $ax^2 + bx + b = 0$ and $ax^2 + ax + b = 0$ have real roots. If the product of one root of the equation $ax^2 + bx + b = 0$ and one root of the equation $ax^2 + ax + b = 0$ is 1, what is $a + b$?

   (A) -2 (B) -1 (C) 2 (D) 3
   (E) None of the answers (A) through (D) is correct.

5. Find the number of all ordered pairs of positive integers (x, y) such that $20x + 18y = 2018$.

   (A) 8 (B) 9 (C) 10 (D) 11 (E) 12

6. Let $p(x) = x^4 + ax^3 + bx^2 + cx + d$ be a polynomial with real coefficients. If $1 + i$ and i are roots of the polynomial $p(x)$, find $a + b + c + d$.

   (A) -1 (B) 0 (C) 1 (D) 2 (E) 3

7. Let M be a point on the side $\overline{BC}$ in the triangle ABC. The line passing through M and parallel to $\overline{AB}$ intersects the side $\overline{AC}$ at a point L. The line passing through M and parallel to $\overline{AC}$ intersects the side $\overline{AB}$ at a point K. The areas of the triangles KBM and LMC are 4 and 9, respectively. Find the area of the quadrilateral $AKML$.

   (A) 5 (B) 6 (C) 10 (D) 12 (E) 13

8. In the set $A = \{3, 6, 9, 10, n\}$ the element n is an integer not equal to any of the other four elements of the set A. If the median of the elements of the set A is equal to the mean of the elements of the set A, find the sum of all possible values of n.

(A) 9 (B) 19 (C) 24 (D) 26 (E) None of the answers (A) through (D) is correct.

9. What is the product of the real solutions of the equation $(x+1)(x+2)(x+3)(x+4) = 3$?

(A) -3 (B) 3 (C) 10 (D) 21 (E) None of the answers (A) through (D) is correct.

10. What is the largest number of acute angles that a convex heptagon (7-sided polygon) can have?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

11. Two spheres with radii 1 and 2 are inscribed in a cone. The larger sphere touches the base and the lateral surface of the cone, and the smaller sphere touches the lateral surface of the cone and the larger sphere. Find the volume of the cone.

(A) $\frac{64\pi}{3}$ (B) $\frac{48\sqrt{3}\pi}{3}$ (C) $18\sqrt{3}\pi$ (D) 24π (E) None of the answers (A) through (D) is correct.

12. Determine the sum of all values of the integer a such that the roots of the equation $x^2 - 2ax - (a+3) = 0$ are integers.

(A) -2 (B) -1 (C) 1 (D) 3 (E) 4

13. The sides of a triangle have length 11, 15, and x, where x is an integer. For how many values of x is the triangle obtuse?

(A) 6 (B) 7 (C) 12 (D) 13 (E) 17

14. Let a and b be the number of digits in 2^{2018} and 5^{2018} respectively. Find $a + b$.

(A) 2017 (B) 2018 (C) 2019 (D) 2020 (E) 2021

15. A closed right circular cylinder has an integer radius and an integer height. The numerical value of its volume is five times the numerical value of its surface area. Determine how many distinct cylinders satisfy this property.

(A) 5 (B) 6 (C) 8 (D) 9 (E) 10

16. If a number is selected at random from the set of five-digit numbers in which the sum of the digits is equal to 43, what is the probability that this number is divisible by 11?

(A) $\frac{2}{5}$ (B) $\frac{1}{5}$ (C) $\frac{1}{6}$ (D) $\frac{1}{11}$ (E) None of the answers (A) through (D) is correct.

17. How many real solutions are there for the equation

$$2^{-|x|} = \frac{1}{2\sqrt{2}}(|x+1| - |x-1|)?$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

18. Find $\sum_{n=1}^{2018} a_n$ if $a_1, a_2, a_3, \ldots$ is a sequence given by

$$a_1 = \frac{1}{2}, \; a_n = \frac{a_{n-1}}{2na_{n-1}+1} \text{ for all integers } n > 1.$$

(A) $\frac{2017}{2018}$ (B) $\frac{2018}{2017}$ (C) $\frac{2018}{2019}$ (D) $\frac{2019}{2018}$ (E) None of the answers (A) through (D) is correct.

19. Let a and b $(a > b)$ be the legs of a right triangle ABC such that $\log \frac{a-b}{\sqrt{2}} = \frac{1}{2}(\log a + \log b)$. Find the absolute value of the difference of the acute angles, in degrees, of $\triangle ABC$.

(A) $18°$ (B) $30°$ (C) $36°$ (D) $48°$ (E) $60°$

20. Determine the minimum value of the function

$$f(x,y,z) = \frac{x^2+y^2+z^2}{xy+yz}, \; x > 0, y > 0, z > 0.$$

(A) 1 (B) $\frac{3}{2}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$ (E) None of the answers (A) through (D) is correct.

PART II: 10 INTEGER ANSWER PROBLEMS

1. Find $(\cot 1° - \tan 1°)\tan 2°$.

2. Find the number of positive integers n such that $n+2$ divides n^5+2.

3. Determine the number of all real numbers x for which the numbers $\sqrt{x^2+2x+1}$, $\frac{x^2+3x-1}{3}$, $x-1$, in the given order, are consecutive members of an arithmetic progression.

4. Three times Andrew's age plus Bekir's age equals twice Claudio's age. Double the cube of Claudio's age is equal to the cube of Bekir's age added to three times the cube of Andrew's age. Their respective ages are relatively prime to each other. How old is Bekir?

5. Determine the positive integer n such that each of the digits $0, 1, 2, \ldots, 9$ shows up exactly once in either n^3 or n^4, but not in both, i.e. if a digit is used in n^3, then it is not used in n^4 and vice versa.

6. Find $\sum_{n=1}^{2018} a_n$, if $a_1, a_2, a_3, \ldots$ is a sequence of integers that satisfy

$$1 + \sum_{d|n} (-1)^{\frac{n}{d}} a_d = 0, \quad \text{for } n = 1, 2, 3 \ldots.$$

7. Let ABC be a triangle such that $AC = BC$ and $\angle ACB = 80^\circ$. Let M be a point inside $\triangle ABC$ such that $\angle MBA = 30^\circ$ and $\angle MAB = 10^\circ$. In degrees, find $\angle AMC$.

8. Find the positive integer n for which $\frac{20^n+18^n}{n!}$ has maximum value.

9. Determine the number of real solutions of the equation

$$x^2 - \lfloor x^2 \rfloor = (x - \lfloor x \rfloor)^2$$

that are in the interval $[1, 100]$. ($\lfloor a \rfloor$ denotes the largest integer not exceeding a.)

10. Let a, b, c, d, e and f be real numbers whose sum is 10 and

$$(a-1)^2 + (b-1)^2 + (c-1)^2 + (d-1)^2 + (e-1)^2 + (f-1)^2 = 6.$$

If the maximum value of f is $\frac{p}{q}$, where p and q are relatively prime positive integers, find $p+q$.

The following problem will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test.

TIE BREAKER PROBLEM

Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined by

$$f(n) = \begin{cases} n - 10, & \text{if } n > 100 \\ f(f(n+11)), & \text{if } n \leq 100 \end{cases}$$

Find $f(50)$.