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NC STATE MATHEMATICS CONTEST
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PART I: 20 MULTIPLE CHOICE PROBLEMS

1. A survey of 100 recent college graduates was made to determine their mean salary. The mean salary found was $45,000. It turns out that one of the alumnus incorrectly answered the survey. He said he earns $35,000 when in fact he earns $53,000. What is the actual mean salary of the 100 graduates?

   (A) $45,150$ (B) $45,165$ **(C)** $45,180$ (D) $45,200$
   (E) None of the answers (A) through (D) is correct.

   Solution: Answer: (C). The actual mean is $\frac{1}{100}(45,000 \cdot 100 + 53,000 - 35,000) = 45,180$.

2. The areas of three faces of a rectangular parallelepiped are 18, 40, and 80. Find its volume.

   (A) 220 (B) 228 (C) 230 **(D)** 240 (E) None of the answers (A) through (D) is correct.

   Solution: Answer: (D). Let a, b, and c be the lengths of the sides on the rectangular parallelepiped such that $ab = 18$, $bc = 40$, and $ac = 80$. If we multiply the last three equations we get $a^2b^2c^2 = 57600$. Hence, the volume of the parallelepiped is $V = abc = \sqrt{a^2b^2c^2} = 240$.

3. Find the sum of the digits of $10^{2017} - 2017$.

   (A) $18,135$ **(B)** $18,144$ (C) $18,149$ (D) $18,153$ (E) None of the answers (A) through (D) is correct.

   Solution: Answer: (B). The number $10^{2017} - 2017$ has form $99\ldots97983$ where the number of 9s before the digit 7 is 2013. The sum of the digits of $10^{2017} - 2017$ is $9 \cdot 2013 + 7 + 9 + 8 + 3 = 18,144$.

4. A box contains three red, six blue, and four yellow balls. If two balls are selected at random, what is the probability that they are both yellow, given that they are the same color?

   (A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$ (E) None of the answers (A) through (D) is correct.

   Solution: Answer: (A). Let Y be the event that the two balls are yellow, and S be the event that the two balls are the same color. Then $P(Y) = \binom{4}{2} = 6$ and $P(S) = \binom{4}{2} + \binom{3}{2} + \binom{6}{2} = 6+3+15 = 24$. Hence the probability that both balls are yellow provided that they are the same color is $P(Y|S) = \frac{P(Y)}{P(S)} = \frac{1}{4}$.

5. Calculate $(\log_3 5 + \log_9 25 + \log_{27} 125 + \cdots + \log_{3^n} 5^n) \log_{25} \sqrt[2n]{27}$.

   (A) $\frac{3}{5}$ (B) $(\log_3 5)^n$ (C) $\frac{9}{5}$ **(D)** $\frac{3}{4}$ (E) None of the answers (A) through (D) is correct.

   Solution: Answer: (D). We will use the property $\log_{a^n} b^n = \log_a b$. Then $(\log_3 5 + \log_9 25 + \log_{27} 125 + \cdots + \log_{3^n} 5^n) \log_{25} \sqrt[2n]{27} = n(\log_3 5)(\log_{25} \sqrt[2n]{27}) = \frac{n \log 5}{\log 3} \cdot \frac{3 \log 3}{4n \log 5} = \frac{3}{4}$.

6. The nonzero integers x, y, and z, in the given order, are three consecutive terms of a geometric progression, while the numbers x, $2y$, and $3z$, in the given order, are three consecutive terms of an arithmetic progression. Find the sum of all possible ratios of the geometric progression.

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) 1 **(D)** $\frac{4}{3}$ (E) None of the answers (A) through (D) is correct.

Solution: Answer: (D). Let q be the ratio of the geometric progression x, y, and z, and let d be the difference of the arithmetic progression x, $2y$, and $3z$. Then $y = xq$, $z = xq^2$, and $2y - x = 3z - 2y$. By substituting the first two equations in the third one we get $x(3q^2 - 4q + 1) = 0$. Since $x \neq 0$, we get $q = 1$ or $q = \frac{1}{3}$. Thus, the sum of all possible ratios of the geometric progression is $\frac{4}{3}$.

7. Find the sum of all real numbers a for which the equation $2x^2 + ax + 5x + 7 = 0$ has only one solution.

(A) -10 (B) 0 (C) 10 (D) 31 (E) None of the answers (A) through (D) is correct.

Solution: Answer: (A). The quadratic equation has only one real solution if its discriminant is 0, i.e. $D = (a+5)^2 - 56 = 0$. Then $a^2 + 10a - 31 = 0$. By Viete's formulas, the sum of the roots of the last equation is -10.

8. The lengths of the heights in a triangle are 12, 15, and 20. Find the area of the triangle.

(A) 120 **(B)** 150 (C) 180 (D) 240 (E) None of the answers (A) through (D) is correct.

Solution: Answer: (B). Let a, b, and c be the lengths of the sides of the triangle and let $h_a = 12$, $h_b = 15$, and $h_c = 20$ be the lengths of its altitudes. From $A = \frac{ah_a}{2} = \frac{bh_b}{2} = \frac{ch_c}{2}$ we get $a : b = h_b : h_a$, $b : c = h_c : h_b$, and $a : c = h_c : h_a$. Then $b = \frac{4}{5}a$ and $c = \frac{3}{5}a$. The area of the triangle is $A = \frac{ah_a}{2} = 6a$. Let $s = \frac{1}{2}(a+b+c) = \frac{6}{5}a$. Using Heron's formula we get $A = \sqrt{s(s-a)(s-b(s-c)} = \frac{6a^2}{25}$. Thus $6a = \frac{6a^2}{25}$. Since $a > 0$, we get $a = 25$ and $A = 150$.

9. How many six-digit numbers can be formed using the digits 1,2, and 3 that do not contain two consecutive 1's?

(A) 224 (B) 256 (C) 416 **(D)** 448 (E) None of the answers (A) through (D) is correct.

Solution: Answer: (D). If zero 1s are used, then there are $2^6 = 64$ numbers. If one 1 is used, then there are $\binom{6}{1}2^5 = 192$ such numbers. If two 2s are used, then we have $2^4 = 16$ ways for four of the digits that are not 1s; there five position where to put the two 1s; hence there are $16\binom{5}{2} = 160$ numbers in this case. If three 1s are used, we have $2^3 = 8$ ways for the other three digits (non 1s), and $\binom{4}{3} = 4$ for the three 1s; thus there are 32 such numbers. We cannot use four 1s. Therefore, there are $64 + 192 + 160 + 32 = 448$ such numbers.

10. Find the product of the real solutions of the equation $1 + x^2 - x^4 = x^5 - x^3 - x$.

(A) -1 (B) $\frac{7}{2}$ **(C)** $\frac{1+\sqrt{5}}{2}$ (D) $-\frac{1+\sqrt{5}}{2}$ (E) None of the answers (A) through (D) is correct.

Solution: Answer: (C). The given equation is equivalent to $(1+x)(1+x^2-x^4) = 0$. Hence $x = -1$ or $1 + x^2 - x^4 = 0$. From the second equation we get $x^2 = \frac{1\pm\sqrt{5}}{2}$. Since $\frac{1-\sqrt{5}}{2} < 0$, the real solutions of $1 + x^2 - x^4 = 0$ are $\pm\sqrt{\frac{1+\sqrt{5}}{2}}$. The product of all real solutions of the given equation is $\frac{1+\sqrt{5}}{2}$.

11. Compute the product

$$\left(1-\frac{4}{1}\right)\left(1-\frac{4}{3^2}\right)\left(1-\frac{4}{5^2}\right)\left(1-\frac{4}{7^2}\right)\cdots\left(1-\frac{4}{99^2}\right)$$

(A) $-\frac{101}{\cdot 99}$ (B) $-\frac{101}{33}$ (C) $-\frac{99}{97}$ (D) $-\frac{101}{97}$
(E) None of the answers (A) through (D) is correct.

Solution: Answer: (A). $\left(1-\frac{4}{1}\right)\left(1-\frac{4}{3^2}\right)\left(1-\frac{4}{5^2}\right)\left(1-\frac{4}{7^2}\right)\cdots\left(1-\frac{4}{99^2}\right)$
$=\left(1-\frac{2}{1}\right)\left(1+\frac{2}{1}\right)\left(1-\frac{2}{3}\right)\left(1+\frac{2}{3}\right)\left(1-\frac{2}{5}\right)\left(1+\frac{2}{5}\right)\left(1-\frac{2}{7}\right)\left(1+\frac{2}{7}\right)\cdots\left(1-\frac{2}{97}\right)\left(1+\frac{2}{97}\right)\left(1-\frac{2}{99}\right)\left(1+\frac{2}{99}\right)$
$=-1\cdot 3\cdot\frac{1}{3}\cdot\frac{5}{3}\cdot\frac{3}{5}\cdot\frac{7}{5}\cdot\frac{5}{7}\cdots\frac{97}{95}\cdot\frac{95}{97}\cdot\frac{99}{97}\cdot\frac{97}{99}\cdot\frac{101}{99}=-\frac{101}{99}$

12. Let ABC be a right triangle and the lengths of its legs are the roots of the equation $ax^2+bx+c=0$. Find the area of the circumscribed circle of the triangle ABC.

(A) $\pi\frac{b^2-2ac}{4a^2}$ (B) $\pi\frac{b^2-4ac}{4a^2}$ (C) $\pi\frac{b^2-2ac}{2a^2}$ (D) $\pi\frac{b^2+2ac}{4a^2}$
(E) None of the answers (A) through (D) is correct.

Solution: Answer: (A). Let u and v be the length of the legs of the triangle ABC and w be the length of its hypotenuse. Then $u+v=-\frac{b}{a}$ and $uv=\frac{c}{a}$. Now we have

$$w^2=u^2+v^2=(u+v)^2-2uv=\left(\frac{b}{a}\right)^2-2\frac{c}{a}=\frac{b^2-2ac}{a^2}.$$

Thus $\left(\frac{w}{2}\right)^2=\frac{b^2-2ac}{4a^2}$. Since the radius if the circumscribed circle is $R=\frac{w}{2}$, we have $A=\pi R^2=\pi\frac{b^2-2ac}{4a^2}$.

13. Let a be a positive real number. Find the sum

$$\log_2 a\log_4 a+\log_4 a\log_8 a+\log_8 a\log_{16} a+\cdots+\log_{2^{n-1}} a\log_{2^n} a$$

(A) $\log_2^2 a\left(1-\frac{1}{n+1}\right)$ **(B)** $\log_2^2 a\left(1-\frac{1}{n}\right)$ (C) $\log_2^2 a$ (D) $\frac{\log_2^2 a}{n}$
(E) None of the answers (A) through (D) is correct.

Solution: Answer: (B). Since $\log_{2^k} a=\frac{1}{k}\log_2 a$ for every $k=1,2,\dots,n$, we get

$$\log_{2^{k-1}} a\log_{2^k} a=\frac{1}{(k-1)k}\log_2^2 a\quad\text{for } k=2,3,\dots,n.$$

Then

$$\sum_{k=2}^{n}\log_{2^{k-1}} a\log_{2^k} a=\sum_{k=2}^{n}\frac{\log_2^2 a}{(k-1)k}=\log_2^2 a\sum_{k=2}^{n}\frac{1}{(k-1)k}=\log_2^2 a\sum_{k=2}^{n}\left(\frac{1}{k-1}-\frac{1}{k}\right)=\log_2^2 a\left(1-\frac{1}{n}\right).$$

14. Let a, b, and c be positive integer numbers such that $\frac{a\sqrt{3}+b}{b\sqrt{3}+c}$ is a rational number. Then $\frac{a^2+b^2+c^2}{a+b+c}$ is equal to

(A) $a+b-c$ **(B)** $a-b+c$ (C) $-a+b+c$ (D) $a-b-c$
(E) None of the answers (A) through (D) is correct.

Solution: Answer: (B). We first have $\frac{a\sqrt{3}+b}{b\sqrt{3}+c} = \frac{a\sqrt{3}+b}{b\sqrt{3}+c} \cdot \frac{b\sqrt{3}-c}{b\sqrt{3}-c} = \frac{(3ab-bc)+(b^2-ac)\sqrt{3}}{3b^2-c}$. Since $\frac{a\sqrt{3}+b}{b\sqrt{3}+c}$ is a rational number, we have $b^2 = ac$. Then

$$a^2+b^2+c^2 = a^2+ac+c^2 = (a+c)^2 - ac = (a+c)^2 - b^2 = (a+c-b)(a+c+b).$$

Hence $\frac{a^2+b^2+c^2}{a+b+c} = a+c-b$.

15. Let $ABCD$ be a trapezoid such that $\overline{AB} \parallel \overline{CD}$, $BC = CD = 7$, $AD = 8$, and $BD \perp AD$. Find the length of AB.

(A) 11 (B) 11 (C) 13 **(D)** 14 (E) None of the answers (A) through (D) is correct.

Solution: Answer: (D). Let F be a point on the side $\overline{AB}$ such that $\overline{CF} \parallel \overline{AD}$. Since the triangle BCD is isosceles, $BD \perp AD$, and $\overline{CF} \parallel \overline{AD}$, we get that $\overline{CF}$ is a bisects the angle BCD. Now we have that $\angle BCF = \angle FCD = \angle BFC$ which implies that th triangle BCF is isosceles, i.e. $FB = BC = 7$. Since $AF = CD = 7$, we get $AB = 14$.

16. How many triples of positive integers (x, y, z) satisfy the equations $x^2+y-z = 100$ and $x+y^2-z = 124$?

(A) 0 **(B)** 1 (C) 2 (D) 3 (E) 4

Solution: Answer: (B). Subtracting the first equation from the second equation we get $x+y^2-x^2-y = 24$, which implies that $(y-x)(y+x-1) = 24$. Since $y-x < y+x-1$, we have the following systems: $y-x = 1, y+x-1 = 24$; $y-x = 2, y+x-1 = 12$; $y-x = 3, y+x-1 = 8$; and $y-x = 4, y+x-1 = 6$. Only the first and the third system have integer solutions: $(3, 6)$ and $(12, 13)$. The pair $(3, 6)$ implies that z is negative. Hence, the only solution of the system is the triple $(12, 13, 57)$.

17. A regular pentagon is inscribed in a circle of radius 1. Let a and d be the lengths of the side and the diagonal of the pentagon. Find a^2+d^2.

(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) 4 **(D)** 5
(E) None of the answers (A) through (D) is correct.

Solution: Answer: (D). Let the regular pentagon $ABCDE$ be inscribed in a circle of radius 1 and center O, let a and d be the lengths of its side and diagonal, respectively. Applying the Law of Cosine for the triangle AOB we get $a^2 = 1^2+1^2-2\cos 72° = 2-2\cos 72°$. Applying the Law of Cosine for the triangle AOC we get $d^2 = 1^2+1^2-2\cos 144° = 2+2\cos 36°$. Then

$$a^2+d^2 = 4+2(\cos 36° - \cos 72°) = 4+4\sin 18° \sin 54° = 4+2\frac{2\cos 18° \sin 18° \sin 54°}{\cos 18°} =$$

$$= 4+\frac{2\sin 36° \cos 36°}{\cos 18°} = 4+\frac{\sin 72°}{\cos 18°} = 4+1 = 5.$$

18. Evaluate the product

$$(1-\cot 1°)(1-\cot 2°)(1-\cot 3°)\cdots(1-\cot 44°).$$

(A) $\left(\frac{\sqrt{2}}{2}\right)^{44}$ (B) $\left(\frac{\sqrt{3}}{2}\right)^{44}$ (C) 3^{22} (D) $(\sqrt{2})^{22}$ **(E)** 2^{22}

Solution: Answer: (E).
$(1-\cot 1^\circ)(1-\cot 2^\circ)(1-\cot 3^\circ)\cdots(1-\cot 44^\circ) = \left(1-\frac{\cos 1^\circ}{\sin 1^\circ}\right)\left(1-\frac{\cos 2^\circ}{\sin 2^\circ}\right)\cdots\left(1-\frac{\cos 44^\circ}{\sin 44^\circ}\right)$ $= \left(\frac{\sin 1^\circ - \cos 1^\circ}{\sin 1^\circ}\right)\left(\frac{\sin 2^\circ - \cos 2^\circ}{\sin 2^\circ}\right)\cdots\left(\frac{\sin 44^\circ - \cos 44^\circ}{\sin 44^\circ}\right)$. Using the identity $\sin\alpha - \cos\alpha = \sqrt{2}\sin(\alpha - 45^\circ)$ we get $\left(\frac{\sin 1^\circ - \cos 1^\circ}{\sin 1^\circ}\right)\left(\frac{\sin 2^\circ - \cos 2^\circ}{\sin 2^\circ}\right)\cdots\left(\frac{\sin 44^\circ - \cos 44^\circ}{\sin 44^\circ}\right) = \frac{\sqrt{2}\sin(1^\circ-45^\circ)\sqrt{2}\sin(2^\circ-45^\circ)\cdots\sqrt{2}\sin(44^\circ-45^\circ)}{\sin 1^\circ \sin 2^\circ \cdots \sin 44^\circ} =$ $= \frac{(\sqrt{2})^{44}(-1)^{44}\sin 44^\circ \sin 43^\circ \cdots \sin 1^\circ}{\sin 1^\circ \sin 2^\circ \cdots \sin 44^\circ} = 2^{22}$.

19. Let $\{a_n\}$ be a finite sequence of real numbers (n is a positive integer number) given with $a_{n+1} = \frac{n+1}{n}a_n + 1$ for $1 \le n \le 2016$, and $a_{2017} = 2017$. Find the sum $a_1 + a_2 + \cdots + a_{2016}$.
(A) $1008 \cdot 2017$ (B) $504 \cdot 2017$ (C) $1009 \cdot 2017$ (D) $\frac{2017 \cdot 2018}{4}$
(E) None of the answers (A) through (D) is correct.

Solution: Answer: (B). From $a_{n+1} = \frac{n+1}{n}a_n + 1$ we get $na_{n+1} - (n+1)a_n = n$. By substituting $n = 1, 2, 3, \ldots, 2016$ in the last equation we get: $1a_2 - 2a_1 = 1$, $2a_3 - 3a_2 = 2$, $3a_4 - 4a_3 = 3$, ..., $2016a_{2017} - 2017a_{2016} = 2016$. Adding all these equations gives $-2(a_1 + a_2 + \cdots + a_{2016}) + 2016a_{2017} = 1 + 2 + 3 + \cdots + 2016 = \frac{2016 \cdot 2017}{2}$. Hence $a_1 + a_2 + \cdots + a_{2016} = -\frac{1}{2}\left(\frac{2016 \cdot 2017}{2} - 2016 \cdot 2017\right) = 504 \cdot 2017$.

20. Let x and y be real numbers such that $3x^2 + 2y^2 \le 6$. Find the greatest value of $2x + y$.
(A) $\frac{3\sqrt{6}}{\sqrt{5}}$ (B) $2\sqrt{3}$ (C) $\sqrt{11}$ (D) $\sqrt{13}$ (E) None of the answers (A) through (D) is correct.

Solution: Answer: (C). We will use Cauchy's Inequality: Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be real numbers. Then $(a_1b_1 + a_2b_2 + \cdot + a_nb_n)^2 \le (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$ and equality holds if and only of $b_i = 0$ for all $i = 1, 2, \ldots, n$ or $a_i = kb_i$ for all $i = 1, 2, \ldots, n$ and some constant k.
$(2x+y)^2 = \left(\frac{2}{\sqrt{3}} \cdot \sqrt{3}x + \frac{1}{\sqrt{2}} \cdot \sqrt{2}y\right)^2 \le \left(\left(\frac{2}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right)\left((\sqrt{3}x)^2 + (\sqrt{2}y)^2\right)$. Hence, $(2x+y)^2 \le \frac{11}{6}(3x^2 + 2y^2)$. Since $3x^2 + 2y^2 \le 6$, we get $(2x+y)^2 \le 11$, i.e. $2x + y \le \sqrt{11}$. The equality holds if $\frac{2}{\sqrt{3}} \cdot \sqrt{2}y = \frac{1}{\sqrt{2}} \cdot \sqrt{3}x$, i.e $3x = 4y$. Then $x = \frac{4}{\sqrt{11}}$ and $y = \frac{3}{\sqrt{11}}$. Therefore, the greatest value is $\sqrt{11}$.

PART II: 10 INTEGER ANSWER PROBLEMS

1. Determine the sum of all positive three-digit integer numbers that give a reminder 2 when divided by 7, a reminder 4 when divided by 9, and reminder 7 when divided by 12.

Solution: Answer: 1497. Let n be a number that satisfies the properties stated in the problem. Then $n + 5$ is divisible by 5, 7, and 12, which implies it is divisible by the least common multiple of 5, 7, and 12. Since $\text{lcm}(5, 7, 12) = 252$, we get that $n + 5 = 252k$ where $k = 1, 2, 3$ since n must be a three digit number. Hence n could be 247, 499, or 751. Their sum is 1497.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(1) = 1$ and $f(x+y) = 3^y f(x) + 2^x f(y)$ for all real numbers x and y. Find $f(4)$.

Solution: Answer: 65. Setting $x=1$ in $f(x+y)=3^y f(x)+2^x f(y)$, we get $f(1+y)=3^y f(1)+2f(y)=3^y+2f(y)$ for all $y\in\mathbb{R}$. Setting $y=1$ in $f(x+y)=3^y f(x)+2^x f(y)$, we get $f(x+1)=3f(x)+2^x f(1)=3f(x)+2^x$ for all $x\in\mathbb{R}$. By renaming the variable y to x, we get that $f(x+1)=3^x+2f(x)$ and $f(x+1)=3f(x)+2^x$. Then $f(x)=3^x-2^x$ and $f(4)=65$.

3. Let x, y, and z be positive integer numbers such that $x<y<z$ and $3^x+3^y+3^z=21897$. Find $x+y+z$.

Solution: Answer: 19. The given equation can be written as $3^x(1+3^{y-x}+3^{z-x})=27\cdot 811$. Since 3^x and 811 are relatively prime, and $1+3^{y-x}+3^{z-x}$ and 27 are also relatively prime, we get $3^x=27$ and $1+3^{y-x}+3^{z-x}=811$. Thus $x=3$. Then the second equation becomes $3^{y-3}+3^{z-3}=810$, i.e. $3^{y-3}(1+3^{z-y})=81\cdot 10$. Since 3^{y-3} and 10 are relatively prime, and $1+3^{z-y}$ and 81 are also relatively prime, we get $3^{y-3}=81$ and $1+3^{z-y}=10$. The second equation fives $y=7$, and from the third equation we get $z=9$. Thus $x+y+z=19$.

4. Let $z_1=\sqrt{a-5}+ai$ and $z_2=2\cos\alpha+3i\sin\alpha$ be two complex numbers where a is a real number such that $a\geq 5$ and $i^2=-1$. Find the minimum value of $\mid z_1-z_2\mid$.

Solution: Answer: 2. Denote $x=\sqrt{a-5}$ and $y=a$. It is clear that $y=x^2+5$. Since $x\geq 0$, we see that the points z_1 lie on the half parabola $y=x^2+5, x\geq 0$. The points z_2 lie on the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$. The closest two points from both sets are the points $(0,5)$ and $(0,3)$. Thus the minimum of $\mid z_1-z_2\mid$ is 2.

5. The number 2^{29} has 9 distinct digits. Which digit is missing?

Solution: Answer: 4. It is an easy observation that an integer number is congruent to the sum of its digits modulo 9. The sum of the digits $0+1+2+\cdots 9=45$ is divisible by 9 and $2^{29}\equiv 2^2(2^3)^9\equiv 2^2(-1)^9\equiv -4\equiv 5(\bmod 9)$. Hence 2^{29} gives a reminder 5 when divided by 9, which implies that the missing digit is 4.

6. Let D be a point on the side $\overline{BC}$ of the isosceles triangle ABC $(AB=BC)$ such that $DC=4BD$. Let E be a point on the side $\overline{AC}$ such that BE is a hight in $\triangle ABC$. Let F be the point of intersection of $\overline{AD}$ and $\overline{BE}$. Find $\frac{EF}{BF}$.

Solution: Answer: 2. Let K be a point on $\overline{BE}$ such that $\overline{DK}||\overline{AC}$. The triangles BCE and BDK are similar. Hence $\frac{BK}{BE}=\frac{DK}{CE}=\frac{BD}{BC}=\frac{1}{5}$; thus $BK=\frac{1}{5}BE$ and $DK=\frac{1}{5}CE$. The triangles DKF and AEF are also similar, and this implies $\frac{KF}{FE}=\frac{KD}{AE}=\frac{KD}{CE}=\frac{1}{5}$. Let $KF=k$. Then $EF=5k$. From the equality $BE=BK+KF+FE$ we get $BK=\frac{3}{2}k$. From $BF=BK+KF=\frac{5}{2}k$. Thus $\frac{EF}{BF}=2$.

7. Find the value of m for which the function $f(x)=\mid x^2-6x\mid -m$ has exactly three x-intercepts.

Solution: Answer: 9. Since $x^2-6x=x(x-6)$, we will consider the given equation on the intervals $(-\infty,0]$, $(0,6)$, and $[6,\infty)$. On the intervals $(-\infty,0]$ and $[6,\infty)$ the given equation has the form $x^2-6x-m=0$ and its solutions are $x=3\pm\sqrt{9+m}$. If $m>0$, then $x=3+\sqrt{9+m}\in[6,\infty)$ and $x=3-\sqrt{9+m}\in(-\infty,0]$. If $m<0$ then $x=3\pm\sqrt{9+m}\in(0,6)$ which is not the interval we are considering. If $m=0$, then the equation has two zeros. On the interval $(0,6)$ the given equation has the form $x^2-6x+m=0$. The given equation will have exactly three zeros if $m>0$ and the equation $x^2-6x+m=0$ has only one zero on $(0,6)$. Hence, the discriminant of $x^2-6x+m=0$ must be 0, i.e. $36-4m=0$. Therefore the given equation has exactly three zeros for $m=9$.

8. Let a and b be positive real numbers such that

$$\log(1+a^2)-\log a-2\log 2=1-\log(100+b^2)+\log b.$$

Find $a+b$.

Solution: Answer: 11. The given equation is equivalent to $\log(1+a^2)+\log(100+b^2)=\log a+\log b+2\log 2+1$. Using properties of logarithmic function we get $\log(1+a^2)(100+b^2)=\log(40ab)$ which implies $(1+a^2)(100+b^2)=40ab$. Since a and b are positive numbers, using the inequality between arithmetic mean and geometric mean we get $1+a^2\geq 2a$ and $100+b^2\geq 20b$ and equality holds when $a=1$ and $b=10$. Hence, if $a\neq 1$ and $b\neq 10$, $(1+a^2)(100+b^2)>40ab$. Therefore, $a=1$, $b=10$, and $a+b=11$.

9. Let a, b, and c be distinct positive integer numbers greater than 1. Let $\frac{m}{n}$ be the maximum of $\left(1+\frac{1}{a}\right)\left(2+\frac{1}{b}\right)\left(3+\frac{1}{c}\right)$, where m and n are relatively prime positive integers. Find $m+n$.

Solution: Answer: 99. If $\min\{a,b,c\}\geq 3$, then $\left(1+\frac{1}{a}\right)\left(2+\frac{1}{b}\right)\left(3+\frac{1}{c}\right)\leq\frac{280}{27}<\frac{91}{8}$. Otherwise $\min\{a,b,c\}\leq 2$ and we consider three cases. If $\min\{a,b,c\}=c=2$, then $a\geq 3$ and $b\geq 3$ and $\left(1+\frac{1}{a}\right)\left(2+\frac{1}{b}\right)\left(3+\frac{1}{2}\right)\leq\left(1+\frac{1}{3}\right)\left(2+\frac{1}{3}\right)\left(3+\frac{1}{2}\right)=\frac{98}{9}<\frac{91}{8}$. If $\min\{a,b,c\}=b=2$, then $a\geq 3$ and $c\geq 3$ and $\left(1+\frac{1}{a}\right)\left(2+\frac{1}{2}\right)\left(3+\frac{1}{c}\right)\leq\left(1+\frac{1}{3}\right)\left(2+\frac{1}{2}\right)\left(3+\frac{1}{3}\right)=\frac{100}{9}<\frac{91}{8}$. If $\min\{a,b,c\}=a=2$, then $b\geq 3$ and $c\geq 4$ or $b\geq 4$ and $c\geq 3$ since b and c are distinct (we could have considered sub-cases similar to this in the previous cases, but it was not necessary). Then $\left(1+\frac{1}{a}\right)\left(2+\frac{1}{b}\right)\left(3+\frac{1}{c}\right)\leq\left(1+\frac{1}{2}\right)\left(2+\frac{1}{3}\right)\left(3+\frac{1}{4}\right)=\frac{91}{8}$, or $\left(1+\frac{1}{a}\right)\left(2+\frac{1}{b}\right)\left(3+\frac{1}{c}\right)\leq\left(1+\frac{1}{2}\right)\left(2+\frac{1}{4}\right)\left(3+\frac{1}{3}\right)=\frac{90}{8}<\frac{91}{8}$. Therefore, the maximum of $\left(1+\frac{1}{a}\right)\left(2+\frac{1}{b}\right)\left(3+\frac{1}{c}\right)$ is $\frac{91}{8}$ which is achieved for $a=2$, $b=3$, and $c=4$. Hence $m=91$, $n=8$ and $m+n=99$.

10. Find the sum of the squares of the real roots of the equation $x^{256}-256^{32}=0$.

Solution: Answer: 8. $x^{256}-256^{32}=x^{(2^8)}-(2^8)^{32}=x^{(2^8)}-2^{(2^8)}=$
$\left(x^{(2^7)}+2^{(2^7)}\right)\left(x^{(2^7)}-2^{(2^7)}\right)=$
$\left(x^{(2^7)}+2^{(2^7)}\right)\left(x^{(2^6)}+2^{(2^6)}\right)\left(x^{(2^6)}-2^{(2^6)}\right)=\cdots$
$\left(x^{(2^7)}+2^{(2^7)}\right)\left(x^{(2^6)}+2^{(2^6)}\right)\cdots(x+2)(x-2)$.
Hence, the sum of the squares of the real roots of the equation is 8.

The following problem, will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test.

TIE BREAKER PROBLEM

Find the sum of all positive integers n such that $n = d_1^2 + d_2^2 + d_3^2 + d_4^2$ where $d_1 < d_2 < d_3 < d_4$ are the four smallest divisors of n.

Solution: Answer: 130. If n is odd, then all of its divisors are odd. Then $d_1^2 + d_2^2 + d_3^2 + d_4^2 \equiv 0 \pmod 4$, which is not possible since we assumed n is odd. Hence n is even, and $d_1 = 1$, $d_2 = 2$. This implies $n \equiv 1 + 0 + d_3^2 + d_4^2 \pmod 4$. If both d_3 and d_4 are even, then $n \equiv 1 \pmod 4$; if both d_3 and d_4 are odd, then $n \equiv 3 \pmod 4$; and if one of d_3 and d_4 is even and the other one is odd, then $n \equiv 2 \pmod 4$. In any case, $n \not\equiv 0 \pmod 4$. Hence $4 \nmid n$ and $(d_3, d_4) = (p, q)$ or $(d_3, d_4) = (p, 2p)$ for some odd primes p and q. If $(d_3, d_4) = (p, q)$, then $n \equiv 3 \pmod 4$, which is not possible since n is an even integer. If $(d_3, d_4) = (p, 2p)$, then $n = 5(1 + p^2)$, and thus $5 \mid n$. Hence $d_3 = 5$ and $d_4 = 10$. Therefore $n = 130$.