The Thirty-ninth Annual State High School Mathematics Contest

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of Science and Mathematics
Durham, NC

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NC STATE MATHEMATICS CONTEST
APRIL 2017

PART I: 20 MULTIPLE CHOICE PROBLEMS

1. A survey of 100 recent college graduates was made to determine their mean salary. The mean salary found was \$45,000. It turns out that one of the alumnus incorrectly answered the survey. He said he earns \$35,000 when in fact he earns \$53,000. What is the actual mean salary of the 100 graduates?
   (A) $45,150$ (B) $45,165$ (C) $45,180$ (D) $45,200$
   (E) None of the answers (A) through (D) is correct.

2. The areas of three faces of a rectangular parallelepiped are 18, 40, and 80. Find its volume.
   (A) 220 (B) 228 (C) 230 (D) 240 (E) None of the answers (A) through (D) is correct.

3. Find the sum of the digits of $10^{2017} - 2017$.
   (A) $18,135$ (B) $18,144$ (C) $18,149$ (D) $18,153$
   (E) None of the answers (A) through (D) is correct.

4. A box contains three red, six blue, and four yellow balls. If two balls are selected at random, what is the probability that they are both yellow, given that they are the same color?
   (A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$ (E) None of the answers (A) through (D) is correct.

5. Calculate $(\log_3 5 + \log_9 25 + \log_{27} 125 + \cdots + \log_{3^n} 5^n) \log_{25} \sqrt[2n]{27}$.
   (A) $\frac{3}{5}$ (B) $(\log_3 5)^n$ (C) $\frac{9}{5}$ (D) $\frac{3}{4}$ (E) None of the answers (A) through (D) is correct.

6. The nonzero integers x, y, and z, in the given order, are three consecutive terms of a geometric progression, while the numbers x, $2y$, and $3z$, in the given order, are three consecutive terms of an arithmetic progression. Find the sum of all possible ratios of the geometric progression.
   (A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) 1 (D) $\frac{4}{3}$ (E) None of the answers (A) through (D) is correct.

7. Find the sum of all real numbers a for which the equation $2x^2 + ax + 5x + 7 = 0$ has only one solution.
   (A) -10 (B) 0 (C) 10 (D) 31 (E) None of the answers (A) through (D) is correct.

8. The lengths of the heights in a triangle are 12, 15, and 20. Find the area of the triangle.
   (A) 120 (B) 150 (C) 180 (D) 240 (E) None of the answers (A) through (D) is correct.

9. How many six-digit numbers can be formed using the digits 1,2, and 3 that do not contain two consecutive 1's?
   (A) 224 (B) 256 (C) 416 (D) 448 (E) None of the answers (A) through (D) is correct.

10. Find the product of the real solutions of the equation $1 + x^2 - x^4 = x^5 - x^3 - x$.

(A) -1 (B) $\frac{7}{2}$ (C) $\frac{1+\sqrt{5}}{2}$ (D) $-\frac{1+\sqrt{5}}{2}$ (E) None of the answers (A) through (D) is correct.

11. Compute the product

$$\left(1-\frac{4}{1}\right)\left(1-\frac{4}{3^2}\right)\left(1-\frac{4}{5^2}\right)\left(1-\frac{4}{7^2}\right)\cdots\left(1-\frac{4}{99^2}\right)$$

(A) $-\frac{101}{99}$ (B) $-\frac{101}{33}$ (C) $-\frac{99}{97}$ (D) $-\frac{101}{97}$
(E) None of the answers (A) through (D) is correct.

12. Let ABC be a right triangle and the lengths of its legs are the roots of the equation $ax^2 + bx + c = 0$. Find the area of the circumscribed circle of the triangle ABC.

(A) $\pi\frac{b^2-2ac}{4a^2}$ (B) $\pi\frac{b^2-4ac}{4a^2}$ (C) $\pi\frac{b^2-2ac}{2a^2}$ (D) $\pi\frac{b^2+2ac}{4a^2}$
(E) None of the answers (A) through (D) is correct.

13. Let a be a positive real number. Find the sum

$$\log_2 a \log_4 a + \log_4 a \log_8 a + \log_8 a \log_{16} a + \cdots + \log_{2^{n-1}} a \log_{2^n} a$$

(A) $\log_2^2 a\left(1-\frac{1}{n+1}\right)$ (B) $\log_2^2 a\left(1-\frac{1}{n}\right)$ (C) $\log_2^2 a$ (D) $\frac{\log_2^2 a}{n}$
(E) None of the answers (A) through (D) is correct.

14. Let a, b, and c be positive integer numbers such that $\frac{a\sqrt{3}+b}{b\sqrt{3}+c}$ is a rational number. Then $\frac{a^2+b^2+c^2}{a+b+c}$ is equal to

(A) $a+b-c$ (B) $a-b+c$ (C) $-a+b+c$ (D) $a-b-c$
(E) None of the answers (A) through (D) is correct.

15. Let $ABCD$ be a trapezoid such that $\overline{AB} \parallel \overline{CD}$, $BC = CD = 7$, $AD = 8$, and $BD \perp AD$. Find the length of AB.

(A) 11 (B) 11 (C) 13 (D) 14 (E) None of the answers (A) through (D) is correct.

16. How many triples of positive integers (x, y, z) satisfy the equations $x^2+y-z = 100$ and $x+y^2-z = 124$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

17. A regular pentagon is inscribed in a circle of radius 1. Let a and d be the lengths of the side and the diagonal of the pentagon. Find $a^2 + d^2$.

(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) 4 (D) 5
(E) None of the answers (A) through (D) is correct.

18. Evaluate the product
$$(1-\cot 1^\circ)(1-\cot 2^\circ)(1-\cot 3^\circ)\cdots(1-\cot 44^\circ).$$
(A) $\left(\frac{\sqrt{2}}{2}\right)^{44}$ (B) $\left(\frac{\sqrt{3}}{2}\right)^{44}$ (C) 3^{22} (D) $(\sqrt{2})^{22}$ (E) 2^{22}

19. Let $\{a_n\}$ be a finite sequence of real numbers (n is a positive integer number) given with $a_{n+1} = \frac{n+1}{n}a_n + 1$ for $1 \le n \le 2016$, and $a_{2017} = 2017$. Find the sum $a_1 + a_2 + \cdots + a_{2016}$.
(A) $1008 \cdot 2017$ (B) $504 \cdot 2017$ (C) $1009 \cdot 2017$ (D) $\frac{2017 \cdot 2018}{4}$
(E) None of the answers (A) through (D) is correct.

20. Let x and y be real numbers such that $3x^2 + 2y^2 \le 6$. Find the greatest value of $2x + y$.
(A) $\frac{3\sqrt{6}}{\sqrt{5}}$ (B) $2\sqrt{3}$ (C) $\sqrt{11}$ (D) $\sqrt{13}$ (E) None of the answers (A) through (D) is correct.

PART II: 10 INTEGER ANSWER PROBLEMS

1. Determine the sum of all positive three-digit integer numbers that give a reminder 2 when divided by 7, a reminder 4 when divided by 9, and reminder 7 when divided by 12.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(1) = 1$ and $f(x + y) = 3^y f(x) + 2^x f(y)$ for all real numbers x and y. Find $f(4)$.

3. Let x, y, and z be positive integer numbers such that $x < y < z$ and $3^x + 3^y + 3^z = 21897$. Find $x + y + z$.

4. Let $z_1 = \sqrt{a-5} + ai$ and $z_2 = 2\cos\alpha + 3i\sin\alpha$ be two complex numbers where a is a real number such that $a \ge 5$ and $i^2 = -1$. Find the minimum value of $| z_1 - z_2 |$.

5. The number 2^{29} has 9 distinct digits. Which digit is missing?

6. Let D be a point on the side $\overline{BC}$ of the isosceles triangle ABC ($AB = BC$) such that $DC = 4BD$. Let E be a point on the side $\overline{AC}$ such that BE is a hight in $\triangle ABC$. Let F be the point of intersection of $\overline{AD}$ and $\overline{BE}$. Find $\frac{EF}{BF}$.

7. Find the value of m for which the function $f(x) =| x^2 - 6x | -m$ has exactly three x-intercepts.

8. Let a and b be positive real numbers such that
$$\log(1 + a^2) - \log a - 2\log 2 = 1 - \log(100 + b^2) + \log b.$$
Find $a + b$.

9. Let a, b, and c be distinct positive integer numbers greater than 1. Let $\frac{m}{n}$ be the maximum of $\left(1+\frac{1}{a}\right)\left(2+\frac{1}{b}\right)\left(3+\frac{1}{c}\right)$, where m and n are relatively prime positive integers. Find $m+n$.

10. Find the sum of the squares of the real roots of the equation $x^{256} - 256^{32} = 0$.

The following problem, will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test.

TIE BREAKER PROBLEM

Find the sum of all positive integers n such that $n = d_1^2 + d_2^2 + d_3^2 + d_4^2$ where $d_1 < d_2 < d_3 < d_4$ are the four smallest divisors of n.