North Carolina Mathematics Level II Contest

2016

1. Consider the equation $x_1 + x_2 + x_3 = 2$ such that $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$. Find the number of integer solutions.

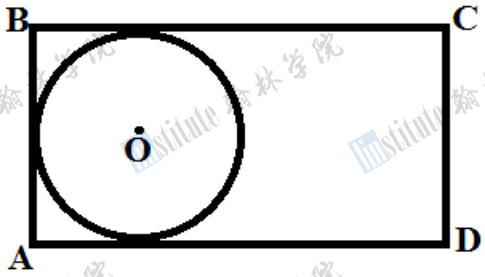
   a. 3 b. 4 c. 6 d. 12 e. 24

2. Evaluate

$$\frac{1}{1^2+1}+\frac{1}{2^2+2}+\frac{1}{3^2+3}+\cdots+\frac{1}{2015^2+2015}+\frac{1}{2016^2+2016}$$

   a. $\frac{1}{9}$ b. $\frac{2013}{2016}$ c. $\frac{2014}{2015}$ d. $\frac{2015}{2016}$ e. $\frac{2016}{2017}$

3. In the figure below, the sides of rectangle *ABCD* are in the ratio $AB:BC = 1:2$. The three sides of the rectangle are tangent to circle *O*. If a point is chosen at random inside the rectangle, what is the probability that it is not inside the circle?



   a. $\frac{4+\pi}{4}$ b. $\frac{4-\pi}{4}$ c. $\frac{4+\pi}{8}$ d. $\frac{8+\pi}{8}$ e. $\frac{8-\pi}{8}$

4. Suppose that the polynomial $(x^3 + mx^2 + nx + 2)(x^2 - x - 1)$ has neither x^3 term nor x^2 term. Compute the ratio $\frac{m}{n}$.

   a. 1 b. $\frac{1}{2}$ c. $\frac{1}{3}$ d. 0 e. undefined

5. Each of the letters of the word BALL is written on an index card and the four index cards are then shuffled. What is the probability that, when the cards are dealt, they spell the word BALL correctly?

   a. $\frac{1}{3}$ b. $\frac{1}{4}$ c. $\frac{3}{4}$ d. $\frac{3}{8}$ e. $\frac{1}{12}$

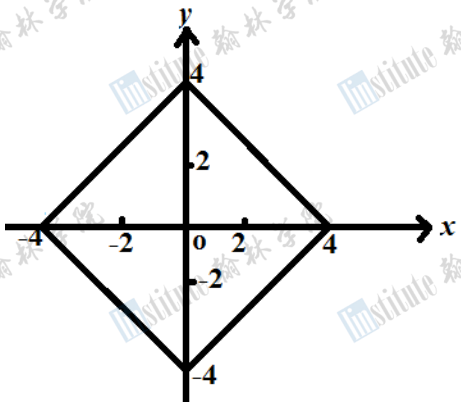
6. Find the number of terms that will appear when the expression $(x + y + 1)^5$ is expanded and the like terms are combined.

   a. 11 b. 17 c. 21 d. 31 e. 77

7. A coin with diameter *d* is placed randomly flat on a paper that contains a grid of squares, *d* units on each side. Find the probability that the coin does not cover a vertex of any square of the grid.

   a. $\frac{2}{\pi+1}$ b. $1-\frac{3\pi}{4}$ c. $\frac{\pi}{16}$ d. $\frac{\pi}{4}$ e. $1-\frac{\pi}{4}$

8. Consider the diagram



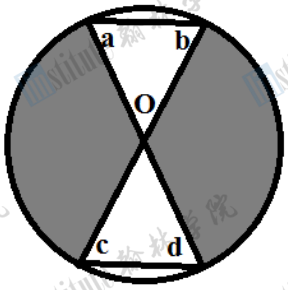
A point (a, b) in the *xy*-plane is called a lattice point if both *a* and *b* are integers. Find the number of lattice points lying strictly inside the given square.

a. 37 b. 34 c. 26 d. 25 e. 24

9. Let $f(x) = x^5 - 2017x^4 + 7102x^2 + 22x + 3015$ and $g(x) = x^4 + 2016x^3 + 2015x^2 + 5102x + 4$. Suppose *a, b, c, d, e, f, g, h* and *i* are all the possible 9 zeros of the two polynomials. (The zeros could be repeated and are not ordered). Let *s* and *t* be the sum and product of these listed zeros, respectively. Find $s + t$.

a. -1101 b. 11011 c. 112059 d. -12059 e. 110011

10. Consider the circle centered at *O*, two chords intersecting at *O*, and where the angles *a, b, c,* and *d* are measured in degrees.



The diameter of the circle *O* is 20 and the area of the shaded region is 80π.
What is the value of $a + b + c + d$?

a. 288 b. 216 c. 240 d. 270 e. 144

11. Find the sum of all the positive odd integers from 1 to 2016.

a. 2016 b. 1008 c. 1008^2 d. 2016^2 e. 1008^3

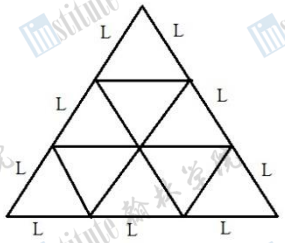
12. A sheet of paper is cut into 5 pieces in the first step. In each of the successive steps, exactly one of the resulting pieces of the previous step is cut into 5 pieces and so on. Which of the following numbers is a possible total number of pieces of paper at some step of this process?

a. 2014 b. 2015 c. 2016 d. 2017 e. none of these

13. A workforce consists of 6 women and 8 men. Suppose 4 workers are chosen at random. If all the workers have an equal chance of being picked, what is the probability that 2 men and 2 women will be selected?
a. $\frac{60}{143}$ b. $\frac{48}{147}$ c. $\frac{32}{153}$ d. $\frac{24}{153}$ e. none of these

14. Consider the family of parabolas $f_p(x) = x^2 + (p+2)x + p$, where $0 < p < 1$. Then all the vertices of the family of parabolas satisfy which of the following conditions?
a. The vertices are situated on the *y*-axis.
b. The vertices are situated on a slant line.
c. The vertices are not situated on a parabola.
d. The vertices are on the *x*-axis.
e. The vertices are in the third quadrant.

15. Compute the sum of the areas of all equilateral triangles in the figure, given that $L = \sqrt[4]{3}$.



a. $9\sqrt{3}$ b. $\frac{45}{2}$ c. $\frac{15\sqrt{3}}{2}$ d. $18\sqrt{3}$ e. 15

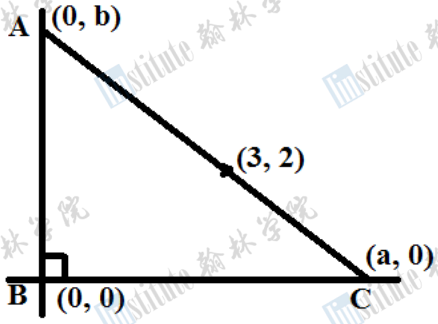
16. Suppose 101 persons arrive at the reception of a conference. If there is exactly one handshake between any two persons, how many handshakes are possible?
a. 101 b. 99 c. 202 d. 5050 e. 10100

17. In the mythical country of Jamais, the natural numbers are defined by the following sequence:
N={1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, 22, 23, 24, 30, 31, 32, 33, 34, 40, ... }.
Suppose $\frac{x}{3} = 2142$ and $y = 3123 + 2314$. Use the Jamaisian number system to compute the value of $x + y$.
a. 23023 b. 12441 c. 13443 d. 23433 e. 10442

18. Find the value of $(a+b)^3$ if $a^3 + b^3 = 5$, $a^2b = 54$ and $ab^2 = 18$.
a. 77 b. 221 c. 35 d. 40 e. 5

19. Suppose that $(-5) + (-3) + (-1) + 1 + 2 + 3 + 4 + 5 + \cdots + n = 181$. Find *n*.
a. 17 b. 18 c. 19 d. 20 e. none of these

20. Consider the right triangle *ABC* with angle *B* a right angle.



The coordinates of the three vertices are $A = (0, b)$, $B = (0,0)$ and $C = (a, 0)$, $a \neq 0$, $b \neq 0$. Find the area of the triangle.

a. $\frac{a^2}{a-3}, a \neq 3$ b. $\frac{a^2}{2-a}, a \neq 2$ c. $\frac{(b-2)a}{2b}$ d. ab e. $\frac{1}{2}a(b-3)$

21. Which of these points is on the line connecting (3, 7) and (5, 4)?

a. (4, 5) b. (–1, 13) c. (8, 11) d. (9, –1) e. none of these

22. Determine the area of a rectangle with a perimeter of 28 cm and a diagonal 10 cm.

a. 48 cm^2 b. 49 cm^2 c. 50 cm^2 d. 160 cm^2 e. 180 cm^2

23. Six coins (three are gold and three are silver) are evenly distributed among three kids. If the process is random, what is the probability that each kid will have a gold and a silver coin?

a. $\frac{1}{20}$ b. $\frac{1}{6}$ c. $\frac{195}{720}$ d. $\frac{1}{3}$ e. $\frac{2}{5}$

24. Carl starts at 7 a.m. for his morning run. His pace is 10 km per hour. Margi starts a half hour later trying to catch up to him on her bike going 30 km per hour. When will she catch up?

a. 7:10 a.m. b. 7:15 a.m. c. 7:40 a.m. d. 7:45 a.m. e. 7:54 a.m.

25. Let *a*, *b* and *c* each represent a distinct digit, and let the three digit number, *abc*, be divisible by both 9 and 11, and the two digit number, *ac*, be divisible by 8. Determine *a*.

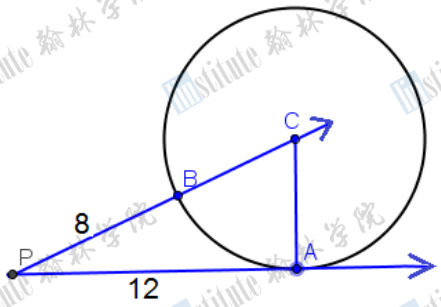
a. 3 b. 4 c. 5 d. 6 e. 7

26. Determine the area of the triangle formed by connecting the points (1, 2), (4, 7) and (6, 5).

a. 6.5 b. 8 c. 10.5 d. 15 e. 16

27. Given a circle centered at *C* and a point *P* outside the circle. Let two rays emanate from *P* such that one of them is tangent to the circle at point *A* and the other passes through *C*, the circle's center. Let *B* be the intersection of the circle and $\overline{PC}$.

If *PA* = 12 and *PB* = 8, what is the diameter of the circle?



a. 5 b. $2\sqrt{20}$ c.10 d. 14 e. none of these

28. Let ϕ be the positive solution to $\phi - \frac{1}{\phi} = 1$. Which of the following is equivalent to ϕ^7?

a. $7\phi + 6$ b. $8\phi - 5$ c. $8\phi + 5$ d. $13\phi + 8$ e. $21\phi + 13$

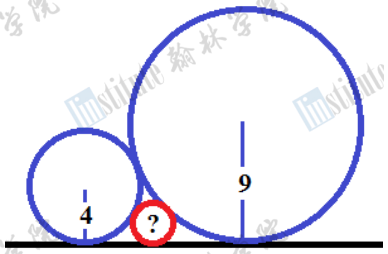
29. If 7^{2016} is divided by 5, what is its remainder?

a. 0 b. 1 c. 2 d. 3 e. 4

30. Peter has to drive 120 miles to get to his destination. He drives 90 mph for the first 30 miles. He then gets a ticket which took 15 minutes to process, and he continues his trip driving at 60 mph until he reaches his destination. What was his average speed for the entire trip?

a. 57.6 mph b. 60 mph c. 61.25 mph d. 67.5 mph e. none of these

31. Three mutually tangent circles are each tangent to a line. If the two larger circles have radii of lengths 4 and 9 what is the length of the third circle's radius?



a. $\frac{4}{9}$ b. $\frac{25}{36}$ c. 1 d. 1.2 e. 1.44

32. If an equilateral triangle has the same area as a regular hexagon, what is the ratio of the hexagon's perimeter to the triangle's perimeter?

a. $\frac{\sqrt{6}}{6}$ b. $\frac{\sqrt{6}}{2}$ c. $\frac{3\sqrt{3}}{2}$ d. $\frac{2\sqrt{2}}{3}$ e. $\frac{\sqrt{6}}{3}$

33. In Charlotte of the 200,000 voters 55% voted for candidate D. Of Asheville's 40,000 voters only 35% voted for candidate D, and in Boone only 30% of 10,000 voters supported "D". Of all voters, what percentage voted for "D"?

a. 50.8% b. 47⅓% c. 44⅔% d. 40% e. none of these

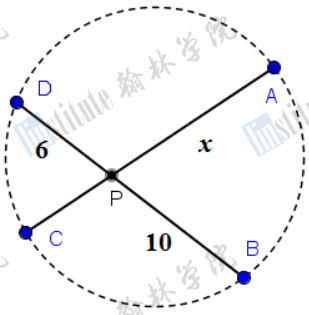
34. Determine the distance between the two intersections of line, $y = 2x - 6$ and parabola, $y = x^2 - 9$.

a. $\sqrt{42}$ b. $2\sqrt{17}$ c. $4\sqrt{5}$ d. 12 e. none of these

35. Consider a circle with chords $\overline{CA}$ and $\overline{DB}$ intersecting at P.

Let $CA = 17$, $DP = 6$, and $PB = 10$. If $CP < AP$, determine the measure of $\overline{AP}$.

a. 5 b. $\frac{17+9\sqrt{21}}{2}$ c. $\frac{17+7\sqrt{3}}{2}$ d. 12 e. none of these



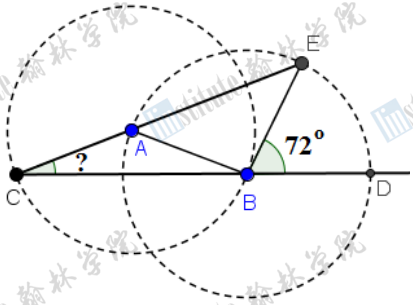
36. If you list all the natural numbers from 1 to 2016, how many do NOT contain the digit "7"?

a. 543 b. 1,315 c. 1,473 d. 1,743 e. none of these

37. A tape 1mm thick and 5 cm wide is wrapped around a 5 cm wide circular core that has a 4 cm diameter. If you wrap the tape 50 times around this core, approximately how long is the tape?

a. 6 m b. 10 m c. 13 m d. 14 m e. 15 m

38. Two circles centered at points *A* and *B* each contain the other's center. Point *C* is on one circle, and points *E* and *D* lie on the other circle. Points *C*, *A* and *E* are collinear as are points *C*, *B* and *D*.
If $\angle EBD = 72°$ determine $\angle ACB$.



a. 18° b. 20° c. 24° d. 30° e. 36°

Problems 39 & 40 address this cross number puzzle. Each cell contains one digit and *k* and *n* are natural numbers. Digits may appear more than once. Solve the puzzle then answer the questions.

Across
1. Sum of all integers from 1 to *n*.
4. k^3
5. A factorial
Down
1. $(n-k)^2$
2. A power of 2
3. Number of 4 letter permutations using the word "ANSWER".

1	2	3
4		
5		

39. What is the value of $n+k$?

a. 24 or less b. 25 c. 27 d. 29 e. 30 or greater

40. Which digit listed below was NOT represented in the puzzle grid?

a. 1 b. 3 c. 4 d. 5 e. 6