The Thirty-eighth Annual State High School Mathematics Contest

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NC STATE MATHEMATICS CONTEST
APRIL 2016

PART I: 20 MULTIPLE CHOICE PROBLEMS

1. The product $\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{n-1}\right)\left(1+\frac{1}{n}\right)$ is equal to

   (A) 1 **(B)** $1+\frac{1}{n}$ (C) -1 (D) $\frac{1}{n}$ (E) None of the answers (A) through (D) is correct.

   Solution: Answer: (B). Notice that $\left(1+\frac{1}{i}\right)\left(1-\frac{1}{i+1}\right)=1$. Thus, the product is equal to $1+\frac{1}{n}$.

2. Find the area of the region above the x-axis and below the graph of $x^2+y^2=1-2y$.

   (A) $\frac{\pi}{2}$ (B) $2-\frac{\pi}{2}$ **(C)** $\frac{\pi}{2}-1$ (D) $\frac{3\pi}{2}+1$ (E) None of the answers (A) through (D) is correct.

   Solution: Answer: (C). The equation $x^2+y^2=1-2y$ is equivalent to $x^2+(y+1)^2=2$ whose graph is a circle with center at $(0,-1)$ and radius $\sqrt{2}$. This circle intersect the x-axis at $(-1,0)$ and $(1,0)$. The area of the region below the circle and above the x-axis is equal to the area of one quarter of a circle with radius $\sqrt{2}$ minus the area of a right triangle with legs $\sqrt{2}$, i.e. $\frac{\pi}{2}-1$.

3. Let x be a real number greater than 1 such that $x-\frac{1}{x}=\sqrt{x}+\frac{1}{\sqrt{x}}$. Determine the value of $x+\frac{1}{x}$.

   (A) $\sqrt{6}$ (B) 4 **(C)** 3 (D) 5 (E) None of the answers (A) through (D) is correct.

   Solution: Answer: (C). Let $a=\sqrt{x}$. Then $a^2-\frac{1}{a^2}=a+\frac{1}{a}$, which is equivalent to $\left(a-\frac{1}{a}\right)\left(a+\frac{1}{a}\right)=a+\frac{1}{a}$. Since $x>1$, we have $a>1$; form the previous equation we get $a-\frac{1}{a}=1$. Hence, $a^2-2+\frac{1}{a^2}=1$, which implies $a^2+\frac{1}{a^2}=3$. Therefore, $x+\frac{1}{x}=3$.

4. Points A and B are on the parabola $y=2x^2+4x-2$. The origin is the midpoint of the line segment joining A and B. Find the length of this line segment.

   (A) $2\sqrt{17}$ (B) 8 (C) $\sqrt{70}$ (D) 9 (E) None of the answers (A) through (D) is correct.

   Solution: Answer: (A). Let A has coordinates (a,b). Then B has coordinates $(-a,-b)$. Both points are on the parabola; hence $b=2a^2+4a-2$ and $-b=2a^2-4a-2$. We get that $a=1$ or $a=-1$. If $a=1$, then $b=4$; if $a=-1$, then $b=-4$. The length of the line segment joining A and B is $2\sqrt{17}$.

5. Find the sum of all 3-digit positive integers that are 34 times the sum of their digits.

   (A) 102 (B) 306 (C) 510 (D) 612 **(E) None of the answers (A) through (D) is correct.**

   Solution: Answer: (E). Let $\overline{abc}$ be a 3-digit number that is 34 times the sum of its digits. Then $100a+10b+c=34(a+b+c)$. Then $22a-11c=8b$, which implies that b is divisible by 11. Since b is a digit, we get $b=0$ and $c=2a$. The 3-digit numbers that satisfy this property are: 102, 204, 306, and 408. Their sum is 1020.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies $f(x+31) = f(31-x)$ for all real numbers x. If f has exactly three real roots, then their sum is:

(A) 0 (B) 31 (C) 62 **(D)** 93 (E) None of the answers (A) through (D) is correct.

> **Solution:** Answer: (D). Notice that the function f has the same value at points symmetric about 31. Since f has exactly three real roots, one of the roots must be 31, and the other two roots must be $31 - a$ and $31 + a$ for some nonzero real number a. The sum of the roots is 93.
>
> **Remark:** There was a mistake on this question during the contest. The asumpotion “f(x+31)=f(31-x)$forallrealnumbersx''wastypedas``f(x+31) = f(x-31)$ for all real numbers x” which does not imply the symmetry discussed in the solution. Therefore, the answer to the question on the contest was (E).

7. Let a, b, and c be three consecutive members of a geometric progression (in the given order). Assume $a > 1, b > 1, c > 1$ and the common ration of the geometric progression is greater than 1. Then $\dfrac{\log_b 3\,(\log_{a^2} c - \log_c \sqrt{a})}{\log_a 9 - 2\log_c 3}$ equals to

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\log_b (c\sqrt{a})$ (D) 3 (E) None of the answers (A) through (D) is correct.

> **Solution:** Answer: (A). Since a, b, and c are members of a geometric progression, in the given order, then $b^2 = ac$. Then $\dfrac{\log_b 3\,(\log_{a^2} c - \log_c \sqrt{a})}{\log_a 9 - 2\log_c 3} = \dfrac{\frac{\log_3 3}{\log_3 b}\left(\frac{\log_3 c}{\log_3 a^2} - \frac{\log_3 \sqrt{a}}{\log_3 c}\right)}{\frac{\log_3 9}{\log_3 a} - 2\frac{\log_3 3}{\log_3 c}} = \dfrac{\frac{1}{\log_3 b}\left(\frac{\log_3 c}{2\log_3 a} - \frac{\frac{1}{2}\log_3 a}{\log_3 c}\right)}{\frac{2}{\log_3 a} - 2\frac{1}{\log_3 c}} =$
> $\dfrac{1}{4}\dfrac{1}{\log_3 b}(\log_3 c + \log_3 a) = \dfrac{1}{4}\dfrac{\log_3 ac}{\log_3 b} = \dfrac{\log_3 b^2}{4\log_3 b} = \dfrac{2\log_3 b}{4\log_3 b} = \dfrac{1}{2}$

8. Determine the number of integer numbers a for which the equation $x^3 - 13x + a = 0$ has three integer roots.

(A) 0 (B) 1 **(C)** 2 **(D)** 4 (E) None of the answers (A) through (D) is correct.

> **Solution:** Answer: (C). Let x_1, x_2, and x_3 be the integer roots of the given equation. Then $x_1 + x_2 + x_3 = 0$, $x_1x_2 + x_2x_3 + x_1x_3 = -13$, and $x_1x_2x_3 = -a$. Then $x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_2x_3 + x_1x_3) = 26$. Since x_1, x_2, x_3 are all integers, their squares are also integer numbers. We get that $\{x_1^2, x_2^2, x_3^2\} = \{0, 1, 25\}$ or $\{x_1^2, x_2^2, x_3^2\} = \{1, 9, 16\}$. If $\{x_1, x_2, x_3\} = \{\pm 0, \pm 1, \pm 5\}$, then the equation $x_1 + x_2 + x_3 = 0$ is not satisfied. If $\{x_1, x_2, x_3\} = \{\pm 1, \pm 3, \pm 4\}$, then there are two possibilities for the roots: $\{1, 3, -4\}$ and $\{-1, -3, 4\}$. Then $a = -12$ or $a = 12$.

9. Adam and David are playing a game on a circular board with n spaces. Both players place their chip at the same starting space. First Adam moves his chip forward five spaces from the starting space, then David moves his chip forward seven, then Adam five, then David seven, and so on. The first player to finish his turn on the starting space wins the game. If n is a random two-digit number, what is the probability that Adam wins?

(A) $\frac{77}{90}$ (B) $\frac{75}{90}$ (C) $\frac{32}{90}$ (D) $\frac{31}{90}$ (E) $\frac{13}{90}$

Solution: Answer: (A). If n is not divisible by neither 5 nor 7, Adam will finish when he moves forward $5n$ spaces and David will finish when he moves forward $7n$ spaces. Hence, if 5 does not divide n and 7 does not divide n, Adam will win.

If n is divisible by 5, but not by 7, then Adam will win; he will finish in $\frac{n}{5}$ moves and David will finish in $7n$ moves. If n is divisible by 7, by similar arguments as in the previous case, David will win. If n is divisible by both 5 and 7, then David will win. Hence, David wins if and only if n is divisible by 7. There are 13 2-digit numbers divisible by 7. Hence, Adam will win in all other cases, $90 - 13 = 77$. The probability that Adam will win is $\frac{77}{90}$.

10. Determine the value of $\sin^3 18° + \sin^2 18°$.

(A) $\frac{1}{4}$ **(B)** $\frac{1}{8}$ (C) $\frac{3\sqrt{3}}{2}$ (D) $\frac{1}{2}$ (E) None of the answers (A) through (D) is correct.

Solution: Answer: (B). $\sin^3 18° + \sin^2 18° = \sin^2 18°(\sin 18° + 1) = \sin^2 18°(\sin 18° + \sin 90°) = = 2\sin^2 18° \sin 54° \cos 36° = 2\sin^2 18° \cos^2 36° = \dfrac{2\sin^2 18° \cos^2 18° \cos^2 36°}{\cos^2 18°} = \dfrac{\sin^2 36° \cos^2 36°}{2\cos^2 18°} = \dfrac{\sin^2 72°}{8\cos^2 18°} = \dfrac{\cos^2 18°}{8\cos^2 18°} = \dfrac{1}{8}$.

11. Let I denote the center of the inscribed circle in the triangle ABC. If one of the triangles AIB, BIC, or CIA is similar to triangle ABC, find the largest angle (in radians) of $\triangle ABC$.

(A) $\frac{5\pi}{7}$ **(B)** $\frac{4\pi}{7}$ (C) $\frac{3\pi}{5}$ (D) $\frac{4\pi}{5}$ (E) None of the answers (A) through (D) is correct.

Solution: Answer: (B). Without loss of generality, assume that $\triangle AIB \sim \triangle ABC$. We first determine the pairs of equal angles of both triangles. The angles of $\triangle AIB$ are $\frac{1}{2}\angle A$, $\frac{1}{2}\angle B$, and $\frac{\pi}{2} + \frac{1}{2}\angle C$. It is clear that $\frac{1}{2}\angle A$ cannot be equal to $\angle A$, $\frac{1}{2}\angle B$ cannot be equal to $\angle B$, and $\frac{\pi}{2} + \frac{1}{2}\angle C$ cannot be equal to $\angle C$. Therefore either $\angle A = \frac{1}{2}\angle B$, $\angle B = \frac{\pi}{2} + \frac{1}{2}\angle C$, $\angle C = \frac{1}{2}\angle A$, or $\angle A = \frac{\pi}{2} + \frac{1}{2}\angle C$, $\angle B = \frac{1}{2}\angle A$, $\angle C = \frac{1}{2}\angle B$. In both cases we have that the largest angle is $\frac{4}{7}\pi$.

12. The value of $\arctan\frac{1}{2} + \arctan\frac{1}{4} + \arctan\frac{1}{13}$ is

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{5}$ (E) None of the answers (A) through (D) is correct.

Solution: Answer: (B) Let $x = \arctan\frac{1}{2}$, $y = \arctan\frac{1}{4}$, and $z = \arctan\frac{1}{13}$. Then $\tan x = \frac{1}{2}$, $\tan y = \frac{1}{4}$, $\tan z = \frac{1}{13}$, and $x, y, z \in \left(0, \frac{\pi}{4}\right)$. Applying the identity $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ twice, we get $\tan(x + y + z) = 1$. Hence, $\arctan\frac{1}{2} + \arctan\frac{1}{4} + \arctan\frac{1}{13} = x + y + z = \arctan 1 = \frac{\pi}{4}$.

13. Let A, B, and C be points on a circle of radius 3. In the triangle ABC, $\angle ACB = 30°$ and $AC = 2$. Find the length of the segment $\overline{BC}$.

(A) $\sqrt{2} + \sqrt{3}$ (B) $\sqrt{2} + 2\sqrt{3}$ **(C)** $2\sqrt{2} + \sqrt{3}$ (D) $2\sqrt{2}$
(E) None of the answers (A) through (D) is correct.

Solution: Answer: (C). Let D be a point on the circle such as AD is a diametar of the circle. Then $AD = 6$, $\angle ADB = 30°$, $\angle ABD = 90°$, and $\angle BAD = 60°$. Since $AD = 6$, we get $AB = 3$ and $BD = 3\sqrt{3}$. Since $\angle ACD = 90°$, by Pythagoren Theorem we get $CD = 4\sqrt{2}$. By Ptolemy's Theorem, we have $AB \cdot CD + BD \cdot AC = AD \cdot BC$. Hence, $BC = 2\sqrt{2} + \sqrt{3}$.

14. Let x and y be nonzero real numbers such that

$$x^2 + y\cos^2\alpha = x\sin\alpha\cos\alpha \text{ and } x\cos 2\alpha + y\sin 2\alpha = 0.$$

Then the relationship between x and y is:

(A) $x^2 + 4y^2 = 0$ (B) $x^2 + 4y = 0$ **(C)** $4x^2 + 4y = 1$ (D) $x^2 - 4y = 1$
(E) None of the answers (A) through (D) is correct.

Solution: Answer: (C). The firs equation is equivalent to $x^2 + \frac{1}{2}y(1+\cos 2\alpha) = \frac{1}{2}x\sin 2\alpha$, which is equivalent to $x\sin 2\alpha - y\cos 2\alpha = 2x^2 + y$. Now we get $\sin 2\alpha = \frac{x(2x^2+y)}{x^2+y^2}$ and $\cos 2\alpha = -\frac{y(2x^2+y)}{x^2+y^2}$. Using the Pythagorean identity $\sin^2(2\alpha) + \cos^2(2\alpha) = 1$ and simplifying, we get $\frac{(2x^2+y)^2}{x^2+y^2} = 1$. Hence, $4x^2 + 4y = 1$.

15. Find the number of the integer solutions (x, y) of the equation $x^2y^3 = 6^{12}$.

(A) 9 (B) 6 **(C) 18** (D) 12 (E) None of the answers (A) through (D) is correct.

Solution: Answer: (C). It is clear that y must be a positive integer and x could be positive or negative integer. Since $6^{12} = 2^{12}3^{12}$ it follows that $x = \pm 2^a 3^b$ and $y = 2^c 3^d$ for some nonnegative integers a, b, c, d. Then $x^2y^3 = 2^{2a+3c}3^{2b+3d} = 2^{12}3^{12}$. Hence, $2a + 3c = 12$ and $2b + 3d = 12$. The solutions of these equations are: $(a, c), (b, d) \in \{(0, 4), (3, 2), (6, 0)\}$. Any of the values for a can be paired with any of the values for b; then the values of c and d are determined. Hence, there 9 pairs of positive integers (x, y) that satisfied the given equation. Since x can be negative as well, the total number of solutions is 18.

16. In a triangle ABC let M be the midpoint of the segment $\overline{AB}$ and N be the midpoint of the segment $\overline{AC}$. Let T be the intersection of $\overline{BN}$ and $\overline{CM}$. Let P be the midpoint the segment $\overline{CT}$ and let Q be the intersection of the lines BP and AC. Determine the value of $\frac{CQ+AN}{NQ}$.

(A) $\frac{4}{3}$ (B) $\frac{5}{3}$ (C) 2 **(D)** $\frac{7}{3}$ (E) None of the answers (A) through (D) is correct.

Solution: Answer: (D). Applying Manelaus' Theorem to $\triangle ACM$ and the line BP we get $\frac{AQ}{QC} \cdot \frac{CP}{PM} \cdot \frac{MB}{BA} = 1$. Sine $\frac{CP}{PM} = \frac{1}{2}$ and $\frac{MB}{BA} = \frac{1}{2}$, we get $AQ = 4QC$. This implies $QC = \frac{AC}{5}$. Since $AN = NC = \frac{AC}{2}$, we have $NQ = \frac{3AC}{10}$. Therefore, $\frac{CQ+AN}{NQ} = \frac{\frac{7AC}{10}}{\frac{3AC}{10}} = \frac{7}{3}$.

17. Find the minimum value of the real function $f(x, y) = x^2 + 2xy + 3y^2 + 2x + 6y + 4$.

(A) -1 (B) 0 **(C) 1** (D) 4 (E) 2

Solution: Answer: (C). $f(x, y) = (x + y + 1)^2 + 2(y + 1)^2 + 1$. Hence, the minimum of f is 1 and it is achieved for $x = 0$ and $y = -1$.

18. Let a, b, and c be real numbers in the interval $(0, 1)$ such that $ab + bc + ca = 1$. Find the largest value of $a + b + c + abc$.

(A) does not have the largest value (B) 2 (C) $\frac{26\sqrt{3}}{9}$ (D) $\frac{28\sqrt{3}}{9}$
(E) None of the answers (A) through (D) is correct.

Solution: Answer: (A). $a+b+c+abc = 1+ab+bc+ca-(1-a)(1-b)(1-c) = 2-(1-a)(1-b)(1-c) \leq 2$. However, 2 is achieved when at least one of a, b, or c is 1, which is not possible. Hence, $a+b+c+abc$ does not have a maximum when $a, b, c \in (0, 1)$ and $ab + bc + ca = 1$.

19. Determine the number of all ordered pairs of prime numbers (p, q) such that $p, q < 100$ and $p+6$, $p+10$, $q+4$, $q+10$, and $p+q+1$ are all prime numbers.

(A) 2 (B) 3 **(C) 4** (D) 5 (E) None of the answers (A) through (D) is correct.

Solution: Answer: (C). Every prime greater than 3 is of form $6k+1$ or $6k-1$ for some positive integer k. Since $p+6$ is prime, we get $p \neq 2$ and $p \neq 3$. Since $p+10$ is prime, we get $p \neq 6k-1$. Hence, $p = 6k+1$ for some positive integer k. If $q > 6$, then q cannot have the form $6s-1$ ($q+10$ must be prime); hence, if $q > 6$, then $q = 6s+1$ for some positive integer s. However, in this case we would get $p+q+1 = 3(2k+2s+1)$, which is not prime. Hence, $q \leq 5$. Clearly $q \neq 2$ and $q \neq 5$. Therefore, $q = 3$. To find p, we are looking for prime numbers of form $6k+1$ such that $p+6$, $p+10$, and $p+4$ are all primes and $1 \leq k \leq 16$. We get four such numbers: 7, 13, 37, 97. Therefore, there four ordered pairs of primes that satisfy the given requirements: $(7, 3)$, $(13, 3)$, $(37, 3)$, and $(97, 3)$.

20. Let a, b, and c be real numbers from the interval $\left(0, \frac{\pi}{2}\right)$ such that $\cos a = a$, $\sin(\cos b) = b$, and $\cos(\sin c) = c$. Order the numbers a, b, and c from the smallest to the largest.

(A) $b < a < c$ (B) $b < c < a$ (C) $c < b < a$ (D) $c < a < b$
(E) None of the answers (A) through (D) is correct.

Solution: Answer: (A). Since $\sin x < x$ on $\left(0, \frac{\pi}{2}\right)$ and since $f(x) = \cos x$ is decreasing on $\left(0, \frac{\pi}{2}\right)$, we have $\sin(\cos x) < \cos x < \cos(\sin x)$. Then $b = \sin(\cos b) < \cos b$, $\cos c < \cos(\sin c) = c$. Hence, $\cos b - b > 0 = \cos a - a > \cos c - c$. Since the function $g(x) = \cos x - x$ is decreasing on $\left(0, \frac{\pi}{2}\right)$ and $g(b) > g(a) > g(c)$, we conclude $b < a < c$.

PART II: 10 INTEGER ANSWER PROBLEMS

1. Three cards each have one of the digits from 1 through 9 written on them; the three digits written on the cards are distinct. When the three cards are arranged in some order, they make a three digit number. The largest number that can be made plus the second largest number that can be made is 1233. What is the largest number that can be made?

Solution: Answer: 621. Let $100a + 10b + c$ be the largest number we can make with the cards, $a > b > c$. The second largest number is $100a + 10c + b$. The sum of the largest and the second largest number is $200a + 11(b + c) = 1233$. Hence, $a \leq 6$. If $a = 6$, then $b + c = 3$. This implies $b = 2$ and $c = 1$. Therefore, the largest number is 621.

2. A closed right circular cylinder has an integer radius and an integer hight. The numerical value of its volume is four times the numerical value of its total surface area (including the top and the bottom). If the numerical value of the smallest possible volume of the cylinder is written as $K\pi$, where K is a positive integer, find the value of K.

Solution: Answer: 3456. Let r and h be the radius and the height of the cylinder. Then $V = \pi r^2 h$ and $A = 2\pi r^2 + 2\pi rh$. From $\pi r^2 h = 2\pi r^2 + 2\pi rh$ we get $rh = 8r + 8h$, which is equivalent to $(r-8)(h-8) = 64$. Since $r \geq 1$ and $h \geq 1$, we get $r - 8 > -8$ and $h - 8 > 8$. Hence, $(r, h) \in \{(9, 72), (10, 40), (12, 24), (16, 16), (24, 12), (40, 10), (72, 9)\}$. The smallest volume of 3456π is obtained for $r = 12$, $h = 24$.

3. Find the smallest integer number n such that $n + 2002$ and $n - 2002$ are perfect squares.

Solution: Answer: 6098. Assume $n + 2002 = x^2$ and $n - 2002 = y^2$ for some nonnegative integer numbers x and y. Then $n = \frac{1}{2}(x^2 + y^2)$ and $x^2 - y^2 = 4004$. The last equality implies $(x - y)(x + y) = 2^2 \cdot 7 \cdot 11 \cdot 13$. Notice that $0 < x - y \leq x + y$ and $x - y$ and $x + y$ are both even. Hence, $(x - y, x + y) \in \{(2, 2002), (14, 286), (22, 182), (26, 154)\}$, which implies $(x, y) \in \{(1002, 1000), (150, 136), (102, 80), (90, 64)\}$. The values of n are: 1002002, 20498, 8402, 6098. The smallest value of n is 6098.

4. Find the smallest positive integer n so that there are exactly 25 integers i satisfying $2 \leq \frac{n}{i} \leq 5$.

Solution: Answer: 80. The given inequality is equivalent to $\frac{n}{5} \leq i \leq \frac{n}{2}$; hence, we are looking for the smallest integer n such that there are exactly 25 integers i that satisfy $\frac{n}{5} \leq i \leq \frac{n}{2}$. Therefore, $\frac{n}{2} - \frac{n}{5} \geq 24$; it is 24 because it is possible that both fractions on the left side of the inequality to be integer numbers. Hence, $n \geq 80$. If $n = 80$, then $\frac{80}{5} = 16$, $\frac{80}{2} = 40$, and there are exactly 25 integer numbers i such that $16 \leq i \leq 40$.

5. A lattice point is a point (x, y) in the coordinate plane with each of x and y an integer. Determine the number of lattice points in the region $|x| + |y| \leq 100$.

Solution: Answer: 20201. The equation $|x| + |y| = 0$ has one solution. Consider the equations $|x| + |y| = k$ for all $1 \leq k \leq 100$. We will consider four cases:

1. Assume $x \geq 0$ and $y \geq 0$. Then the equation $x + y = k$ has $k + 1$ solutions: $(0, k), (1, k-1), \ldots, (k-1, 1), (k, 0)$.
2. Assume $x \geq 0$ and $y < 0$. Then the equation $x - y = k$ has k solutions: $(0, -k), (1, -(k-1)), \ldots, (k-2, -2), (k-1, -1)$.
3. Assume $x < 0$ and $y \geq 0$. Then the equation $-x + y = k$ has k solutions: $(-1, k-1), (-2, k-2), \ldots, (-(k-2), 2), (-(k-1), 1), (-k, 0)$.
4. Assume $x < 0$ and $y < 0$. Then the equation $-x - y = k$ has $k - 1$ solutions: $(-1, -(k-1)), (-2, -(k-2)), \ldots, (-(k-2), -2), (-(k-1), -1)$.

Hence, for each integer $k > 0$, the equation $|x| + |y| = k$ has $4k$ solutions. Therefore, the number of lattice points inside the region $|x| + |y| \leq 100$ is

$$1 + \sum_{k=1}^{100} 4k = 1 + 4 \cdot \frac{100 \cdot 101}{2} = 20,201.$$

6. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Let $\sum_{i=1}^{n} \lfloor \sqrt{i} \rfloor = 217$. Find the value of n.

Solution: Answer: 50. For any positive integers i and j, if $j^2 \le i < (j+1)^2$, then $\lfloor \sqrt{i} \rfloor = j$. Hence, $\sum_{i=j^2}^{(j+1)^2-1} \lfloor \sqrt{i} \rfloor = \sum_{i=j^2}^{(j+1)^2-1} j = j((j+1)^2 - j^2) = j(2j+1)$. Then

$$\sum_{i=1}^{48} \lfloor \sqrt{i} \rfloor = \sum_{i=1}^{3} \lfloor \sqrt{i} \rfloor + \sum_{i=4}^{8} \lfloor \sqrt{i} \rfloor + \sum_{i=9}^{15} \lfloor \sqrt{i} \rfloor + \sum_{i=16}^{24} \lfloor \sqrt{i} \rfloor + \sum_{i=25}^{35} \lfloor \sqrt{i} \rfloor + \sum_{i=36}^{48} \lfloor \sqrt{i} \rfloor = 203.$$

Since $217 - 203 = 14 = 2 \cdot 7$, we get $\sum_{i=1}^{50} \lfloor \sqrt{i} \rfloor = \sum_{i=1}^{48} \lfloor \sqrt{i} \rfloor + \sum_{i=49}^{50} \lfloor \sqrt{i} \rfloor = 203 + 14 = 217$. Therefore, $n = 50$.

7. Find the greatest integer which divides $(a-b)(b-c)(c-d)(d-a)(a-c)(b-d)$ for any integers a, b, c, and d.

Solution: Answer: 12. Let $f(a,b,c,d) = (a-b)(b-c)(c-d)(d-a)(a-c)(b-d)$. Since $f(0,1,2,3) = -12$, the greatest integer that divides the product $(a-b)(b-c)(c-d)(d-a)(a-c)(b-d)$ must be a divisor of 12. Between any four integers at least two are congruent to each other modulo 3, which implies $f(a,b,c,d) \equiv 0 \pmod 3$. We also have that between four integers either two are even and two are odd or at least three have the same parity; either way, $f(a,b,c,d) \equiv 0 \pmod 4$. Hence, $f(a,b,c,d)$ is always divisible by 12. Therefore, the greatest integer which divides $(a-b)(b-c)(c-d)(d-a)(a-c)(b-d)$ is 12.

8. Let $A_0, A_1, \ldots, A_{100}$ be distinct points on one side of an angle and let $B_0, B_1, \ldots, B_{100}$ be distinct points on the other side of the same angle such that $A_0A_1 = A_1A_2 = \cdots = A_{99}A_{100}$ and $B_0B_1 = B_1B_2 = \cdots = B_{99}B_{100}$. Find the area of the quadrilateral $A_{99}A_{100}B_{100}B_{99}$ if the areas of the quadrilaterals $A_0A_1B_1B_0$ and $A_1A_2B_2B_1$ are equal to 5 and 7 respectively.

Solution: Answer: 203. Denote the vertex of the angle by O. Let $r = OA_0$, $a = A_0A_1$, $s = OB_0$, $b = B_0B_1$, $x_0 = Area(OA_0B_0)$, and $x_n = Area(A_{n-1}A_nB_nB_{n-1})$, $1 \le n \le 100$. Denote $r_0 = \frac{a}{r}$ and $s_0 = \frac{b}{s}$. Then

$$\frac{x_n}{x_0} = \frac{Area(OA_nB_n)}{x_0} - \frac{Area(OA_{n-1}B_{n-1})}{x_0} = \frac{(r+na)(s+nb)}{rs} - \frac{(r+(n-1)a)(s+(n-1)b)}{rs}.$$

Hence, $\frac{x_n}{x_0} = (1+nr_0)(1+ns_0) - (1+(n-1)r_0)(1+(n-1)s_0) = r_0 + s_0 + (2n-1)r_0s_0$. This implies $x_n = (r_0 + s_0 + (2n-1)r_0s_0)x_0$ and $x_{n+1} - x_n = 2r_0s_0x_0$. Hence, the sequence $\{x_n\}$ is an arithmetic progression. Since $x_1 = 5, x_2 = 7$, we get that $x_{100} = 5 + 99 \cdot 2 = 203$

9. Let n be an integer with the property that if m is randomly chosen integer from the set $\{1, 2, 3, \ldots, 1000\}$, the probability that m is a divisor of n is $\frac{1}{100}$. If $n \le 1000$, determine the maximum possible value of n.

Solution: Answer: 976. It is clear that n has exactly 10 positive divisors. If the prime factorization of n is $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, then the number of divisors of n is $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1)$. Since 10 can be written as a product of 1 and 10, or 2 and 5, we get that $n = p^9$ or $n = p_1p_2^4$ where p, p_1 and p_2 are prime numbers. The n has form p^9, then the largest possible value is $n = 2^9 = 512$. If n has form

$p_1 p_2^4$, then $p_2 < 5$ since $2 \cdot 5^4 = 1250$. If $p_2 = 2$, then the largest n is $61 \cdot 2^4 = 976$. If $p_2 = 3$, then the largest n is $11 \cdot 3^4 = 891$. Hence, the largest values for n that satisfies the properties describes in the question is 976.

10. Determine the exact number of digits in the decimal expression of 2^{100}.

Solution: Answer: 31. Notice that $2^{10} = 1024 > 10^3$. Then $2^{100} = (2^{10})^{10} > (10^3)^1 0 = 10^{30}$. On the other hand, $2^{13} = 8192 < 10^4$. Then $2^{100} = 2^9 \cdot (2^{13})^7 < 512 \cdot (10^4)^7 = 512 \cdot 10^{28} = 5.12 \cdot 10^{30}$. Hence, $10^{30} < 2^{100} < 5.12 \cdot 10^{30}$. Since both 10^{30} and $5.12 \cdot 10^{30}$ have 31 digits, we conclude that 2^{100} has 31 digits.

The following problem, will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test.

TIE BREAKER PROBLEM

If m and n are any two integer numbers, let $m \circ n$ denote an integer number determined by m and n. Assume that the operation $\circ$ satisfies the following three rules:

(i) $m \circ (n + s) = (m \circ n) - s$ for all integer numbers m, n, s;

(ii) $(n + s) \circ m = (n \circ m) + 2s$ for all integer numbers m, n, s;

(iii) $1 \circ 1 = 1$

Calculate $47 \circ 20$.

Solution: Answer: 74. Let x and y be any integer numbers. Using the identity (i) we get

$$1 \circ y = 1 \circ (1 + (y - 1)) = (1 \circ 1) - (y - 1) = 1 - y + 1 = 2 - y.$$

Using the identity (ii) and the previous identity, we get

$$x \circ y = (1 + (x - 1)) \circ y = (1 \circ y) + 2(x - 1) = 2 - y + 2x - 2 = 2x - y.$$

Hence, $47 * 20 = 2 \cdot 47 - 20 = 74$.