The Thirty-eighth Annual State High School Mathematics Contest

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NC STATE MATHEMATICS CONTEST
APRIL 2016

PART I: 20 MULTIPLE CHOICE PROBLEMS

1. The product $\left(1+\dfrac{1}{2}\right)\left(1-\dfrac{1}{3}\right)\left(1+\dfrac{1}{4}\right)\left(1-\dfrac{1}{5}\right)\cdots\left(1-\dfrac{1}{n-1}\right)\left(1+\dfrac{1}{n}\right)$ is equal to

   (A) 1 (B) $1+\frac{1}{n}$ (C) -1 (D) $\frac{1}{n}$ (E) None of the answers (A) through (D) is correct.

2. Find the area of the region above the x-axis and below the graph of $x^2+y^2=1-2y$.

   (A) $\frac{\pi}{2}$ (B) $2-\frac{\pi}{2}$ (C) $\frac{\pi}{2}-1$ (D) $\frac{3\pi}{2}+1$ (E) None of the answers (A) through (D) is correct.

3. Let x be a real number greater than 1 such that $x-\frac{1}{x}=\sqrt{x}+\frac{1}{\sqrt{x}}$. Determine the value of $x+\frac{1}{x}$.

   (A) $\sqrt{6}$ (B) 4 (C) 3 (D) 5 (E) None of the answers (A) through (D) is correct.

4. Points A and B are on the parabola $y=2x^2+4x-2$. The origin is the midpoint of the line segment joining A and B. Find the length of this line segment.

   (A) $2\sqrt{17}$ (B) 8 (C) $\sqrt{70}$ (D) 9 (E) None of the answers (A) through (D) is correct.

5. Find the sum of all 3-digit positive integers that are 34 times the sum of their digits.

   (A) 102 (B) 306 (C) 510 (D) 612 (E) None of the answers (A) through (D) is correct.

6. Let $f:\mathbb{R}\to\mathbb{R}$ be a function which satisfies $f(x+31)=f(31-x)$ for all real numbers x. If f has exactly three real roots, then their sum is:

   (A) 0 (B) 31 (C) 62 (D) 93 (E) None of the answers (A) through (D) is correct.

7. Let a, b, and c be three consecutive members of a geometric progression (in the given order). Assume $a>1, b>1, c>1$ and the common ration of the geometric progression is greater than 1. Then $\dfrac{\log_b 3\,(\log_{a^2} c-\log_c\sqrt{a})}{\log_a 9-2\log_c 3}$ equals to

   (A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\log_b(c\sqrt{a})$ (D) 3 (E) None of the answers (A) through (D) is correct.

8. Determine the number of integer numbers a for which the equation $x^3-13x+a=0$ has three integer roots.

   (A) 0 (B) 1 (C) 2 (D) 4 (E) None of the answers (A) through (D) is correct.

9. Adam and David are playing a game on a circular board with n spaces. Both players place their chip at the same starting space. First Adam moves his chip forward five spaces from the starting space, then David moves his chip forward seven, then Adam five, then David seven, and so on. The first player to finish his turn on the starting space wins the game. If n is a random two-digit number, what is the probability that Adam wins?

(A) $\frac{77}{90}$ (B) $\frac{75}{90}$ (C) $\frac{32}{90}$ (D) $\frac{31}{90}$ (E) $\frac{13}{90}$

10. Determine the value of $\sin^3 18° + \sin^2 18°$.

(A) $\frac{1}{4}$ (B) $\frac{1}{8}$ (C) $\frac{3\sqrt{3}}{2}$ (D) $\frac{1}{2}$ (E) None of the answers (A) through (D) is correct.

11. Let I denote the center of the inscribed circle in the triangle ABC. If one of the triangles AIB, BIC, or CIA is similar to triangle ABC, find the largest angle (in radians) of $\triangle ABC$.

(A) $\frac{5\pi}{7}$ (B) $\frac{4\pi}{7}$ (C) $\frac{3\pi}{5}$ (D) $\frac{4\pi}{5}$ (E) None of the answers (A) through (D) is correct.

12. The value of $\arctan\frac{1}{2} + \arctan\frac{1}{4} + \arctan\frac{1}{13}$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{5}$ (E) None of the answers (A) through (D) is correct.

13. Let A, B, and C be points on a circle of radius 3. In the triangle ABC, $\angle ACB = 30°$ and $AC = 2$. Find the length of the segment $\overline{BC}$.

(A) $\sqrt{2}+\sqrt{3}$ (B) $\sqrt{2}+2\sqrt{3}$ (C) $2\sqrt{2}+\sqrt{3}$ (D) $2\sqrt{2}$

(E) None of the answers (A) through (D) is correct.

14. Let x and y be nonzero real numbers such that

$$x^2 + y\cos^2\alpha = x\sin\alpha\cos\alpha \text{ and } x\cos 2\alpha + y\sin 2\alpha = 0.$$

Then the relationship between x and y is:

(A) $x^2 + 4y^2 = 0$ (B) $x^2 + 4y = 0$ (C) $4x^2 + 4y = 1$ (D) $x^2 - 4y = 1$

(E) None of the answers (A) through (D) is correct.

15. Find the number of the integer solutions (x, y) of the equation $x^2y^3 = 6^{12}$.

(A) 9 (B) 6 (C) 18 (D) 12 (E) None of the answers (A) through (D) is correct.

16. In a triangle ABC let M be the midpoint of the segment $\overline{AB}$ and N be the midpoint of the segment $\overline{AC}$. Let T be the intersection of $\overline{BN}$ and $\overline{CM}$. Let P be the midpoint the segment $\overline{CT}$ and let Q be the intersection of the lines BP and AC. Determine the value of $\frac{CQ+AN}{NQ}$.

(A) $\frac{4}{3}$ (B) $\frac{5}{3}$ (C) 2 (D) $\frac{7}{3}$ (E) None of the answers (A) through (D) is correct.

17. Find the minimum value of the real function $f(x, y) = x^2 + 2xy + 3y^2 + 2x + 6y + 4$.

(A) -1 (B) 0 (C) 1 (D) 4 (E) 2

18. Let a, b, and c be real numbers in the interval $(0,1)$ such that $ab+bc+ca=1$. Find the largest value of $a+b+c+abc$.
(A) does not have the largest value (B) 2 (C) $\frac{26\sqrt{3}}{9}$ (D) $\frac{28\sqrt{3}}{9}$
(E) None of the answers (A) through (D) is correct.

19. Determine the number of all ordered pairs of prime numbers (p,q) such that $p,q<100$ and $p+6$, $p+10$, $q+4$, $q+10$, and $p+q+1$ are all prime numbers.
(A) 2 (B) 3 (C) 4 (D) 5 (E) None of the answers (A) through (D) is correct.

20. Let a, b, and c be real numbers from the interval $\left(0,\frac{\pi}{2}\right)$ such that $\cos a=a$, $\sin(\cos b)=b$, and $\cos(\sin c)=c$. Order the numbers a, b, and c from the smallest to the largest.
(A) $b<a<c$ (B) $b<c<a$ (C) $c<b<a$ (D) $c<a<b$
(E) None of the answers (A) through (D) is correct.

PART II: 10 INTEGER ANSWER PROBLEMS

1. Three cards each have one of the digits from 1 through 9 written on them; the three digits written on the cards are distinct. When the three cards are arranged in some order, they make a three digit number. The largest number that can be made plus the second largest number that can be made is 1233. What is the largest number that can be made?

2. A closed right circular cylinder has an integer radius and an integer hight. The numerical value of its volume is four times the numerical value of its total surface area (including the top and the bottom). If the numerical value of the smallest possible volume of the cylinder is written as $K\pi$, where K is a positive integer, find the value of K.

3. Find the smallest integer number n such that $n+2002$ and $n-2002$ are perfect squares.

4. Find the smallest positive integer n so that there are exactly 25 integers i satisfying $2\le\frac{n}{i}\le 5$.

5. A lattice point is a point (x,y) in the coordinate plane with each of x and y an integer. Determine the number of lattice points in the region $|x|+|y|\le 100$.

6. Let $\lfloor x\rfloor$ denote the greatest integer less than or equal to x. Let $\sum_{i=1}^{n}\lfloor\sqrt{i}\rfloor=217$. Find the value of n.

7. Find the greatest integer which divides $(a-b)(b-c)(c-d)(d-a)(a-c)(b-d)$ for any integers a, b, c, and d.

8. Let $A_0, A_1, \ldots, A_{100}$ be distinct points on one side of an angle and let $B_0, B_1, \ldots, B_{100}$ be distinct points on the other side of the same angle such that $A_0A_1 = A_1A_2 = \cdots = A_{99}A_{100}$ and $B_0B_1 = B_1B_2 = \cdots = B_{99}B_{100}$. Find the area of the quadrilateral $A_{99}A_{100}B_{100}B_{99}$ if the areas of the quadrilaterals $A_0A_1B_1B_0$ and $A_1A_2B_2B_1$ are equal to 5 and 7 respectively.

9. Let n be an integer with the property that if m is randomly chosen integer from the set $\{1, 2, 3, \ldots, 1000\}$, the probability that m is a divisor of n is $\frac{1}{100}$. If $n \le 1000$, determine the maximum possible value of n.

10. Determine the exact number of digits in the decimal expression of 2^{100}.

The following problem, will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test.

TIE BREAKER PROBLEM

If m and n are any two integer numbers, let $m \circ n$ denote an integer number determined by m and n. Assume that the operation $\circ$ satisfies the following three rules:

(i) $m \circ (n + s) = (m \circ n) - s$ for all integer numbers m, n, s;

(ii) $(n + s) \circ m = (n \circ m) + 2s$ for all integer numbers m, n, s;

(iii) $1 \circ 1 = 1$

Calculate $47 \circ 20$.