State Mathematics Contest Finals, Level II
April 30, 2015

1. Suppose that a sphere is inscribed in a cube (meaning that the sphere touches all six faces of the cube). How many cubic meters is the volume of the sphere if the volume of the cube is 1 cubic meter?

   A. $\frac{\pi}{6}$ B. $\frac{\pi}{3}$ C. $\frac{5\pi}{6}$ D. $\frac{5\pi}{3}$ E. $\frac{10\pi}{3}$

2. For what value of c will $\frac{x}{5}+\frac{x+c}{x+2}=1$ have one solution

   A. 2.5 B. 2.2 C. 3 D. –1 E. –2.8

3. A line with slope 0.5 intersects a circle centered at the origin at points (–5, 0) and (x, y). Determine x.

   A. $2\sqrt{3}$ B. $\sqrt{6}$ C. 3 D. 4 E. $2\sqrt{5}$

4. The expression $\frac{5+\sqrt{3}}{2+\sqrt{3}}$ can be rewritten as $a+b\sqrt{3}$ where a and b are rationals. Determine a.

   A. 13 B. 7 C. 5 D. 2.5 E. 1

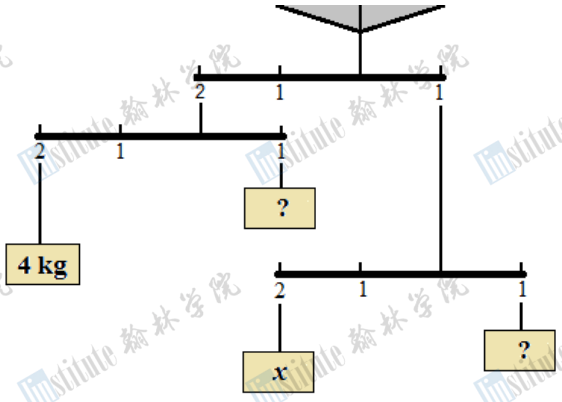
5. If the sum of two numbers is 13 and the sum of their squares is 113, what is their product?

   A. 28 B. 30 C. 36 D. 42 E. 50

6. Consider the mobile displayed on the right. Assume it is perfectly balanced and determine weight x. Remember that sides are balanced if the products of weight and distance from balancing point are equal.

   A. 1 kg B. 4 kg C. 8 kg

   D. 14 kg E. 24 kg



7. The function $f(x) = x^5 - 3x^3 + 5x - 10$ has one real solution. In which interval is that solution?

A. $x \in (-\infty, -2)$ B. $x \in (-2, 0)$ C. $x \in (0, 1)$ D. $x \in (1, 2)$ E. $x \in (2, \infty)$

8. The equation, $5x^2 - 12x = b$, has $x = 3$ as a solution. Determine the second value of x.

A. –0.6 B. –3.6 C. 6 D. 9 E. None of these.

9. Marvin earned an 81 on his essay, a 91 on his midterm, and an 88 on his homework. The only grade missing is his Final Exam. If the essay and midterm are each worth 25% of his grade, and homework is worth 10%, what grade must he make on his final exam to get an average of 90?

A. 94.5 B. 95.5 C. 97.2 D. 98 E. 100

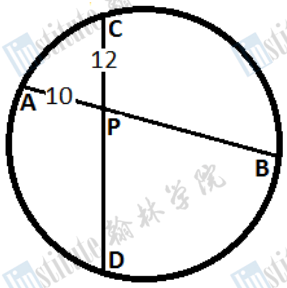
10. Divide the expression 5^{2015} by 7 and determine its remainder.

A. 1 B. 2 C. 3 D. 4 E. 5

11. Let x be defined on all reals. What is the largest value that y can attain if $y = (3x^2 + 4) - (3x - 4)^2$?

A. 2 B. 9⅓ C. 10⅔ D. 12 E. 20

12. Two chords, $\overline{AB}$ and $\overline{CD}$, in a circle intersect at point P as shown. If the length of chord $\overline{CD}$ is 42 and the lengths of segments $\overline{AP}$ and $\overline{CP}$ are 10 and 12 respectively, what is the length of $\overline{AB}$?



A. 25 B. 35 C. 36 D. 40 E. 46

13. Find the sum of all the solutions to the equation: $|3x + 5| - x = 7$.

A. –2 B. 2 C. 4 D. 6 E. None of these

14. Define the two operations, ◊ and ■, on the natural numbers as follows:

◊ = Minimum(k , m) thus if $k < m$ then $k \lozenge m = k$

■ = Maximum(k , m) thus if $k < m$ then $k \blacksquare m = m$

Which of the following statements are true for all x, y and z.

I. $(x \lozenge y) + z = (x + z) \lozenge (y + z)$

II. $(x \lozenge y) \blacksquare z = (x \blacksquare z) \lozenge (y \blacksquare z)$

III. $(x + y) \blacksquare z = (x \blacksquare z) + (y \blacksquare z)$

A. I only B. II only C. I & II D. I & III E. I, II, III

15. Given the operations, ◊ and ■, as defined in the previous problem find the solution set for k.

$(k \lozenge 7) \blacksquare 3 = (k \blacksquare 5) \lozenge 9.$

A. {} B. {1, 2, 3} C. {3, 4, 5} D. {5, 6, 7} E. {7, 8, 9}

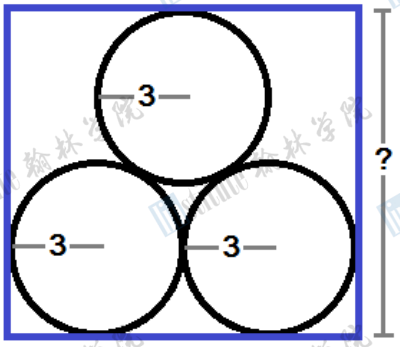
16. What is the area of the region enclosed by $|4x| + |2y| \leq 12$?

A. 24 B. 25 C. 32 D. 36 E. 48

17. In the statement "THE + END = NEAR" each letter represents a digit, different letters represent different digits. In addition "END" is a prime number. Determine the digit that "E" represents.

A. 5 B. 6 C. 7 D. 8 E. 9

18. A rectangle encloses three mutually tangent circles as shown. If each circle has a radius of 3, what is the height of the rectangle?



A. $\sqrt{6}+9$ B. $6\sqrt{3}$ C. $3\sqrt{3}+6$

D. 12 E. 13

19. For the first 30 miles Jack drove at 60 mph, for the next 15 miles his speed was 15 mph, and on the final stretch of 75 miles he drove 75 mph. What was his average speed?

A. 48 mph B. 50 mph. C. 55 mph D. 63.75 mph. E. None of these

20. Let $x = \log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdots \log_{2014} 2015$. Which of the following is true?

A. $3 < x < 4$ B. $4 < x < 5$ C. $5 < x < 6$
D. $6 < x < 7$ E. $x < 3$

21. Let x, y, z be distinct real numbers with $x < 0$. If $x^2 - 2xy + z^2 = 0$ and $yz > x^2$, then the relation between *x*, *y* and *z* is given by:

A. $x < y < z$ B. $y < z < x$ C. $z < x < y$
D. $x < z < y$ E. $y < x < z$

22. Find the real solutions of the equation $3^{2x^2-7x+3} = 4^{x^2-x-6}$

A. 3 only B. 3 and -98 C. 3 and 121
D. 3 and $\frac{1+2\log_3 4}{2-\log_3 4}$ E. 3 and $\frac{1+2\log_4 3}{1-\log_4 3}$

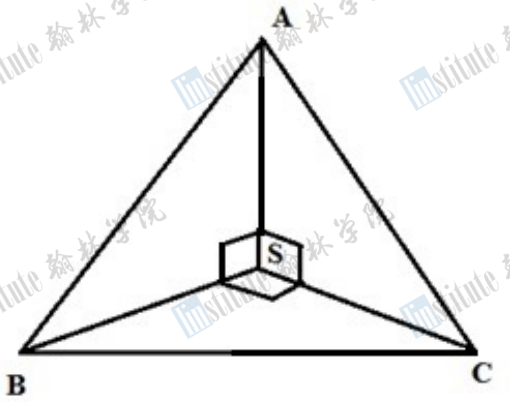
23. Let a, b, c, d be real numbers such that $a + b + c + d = 9$ and $a^2 + b^2 + c^2 + d^2 = 27$. Find the maximum value of *d*.

A. 3.75 B. $3\sqrt{3}$ C. 4.50 D. $3\sqrt{2}$ E. 1.25

24. If the three medians of a triangle *ABC* are 3, 4, and 5 feet, what is the area of the triangle *ABC* in square feet?

A. 8 B. 6 C. 10 D. 12 E. None of these

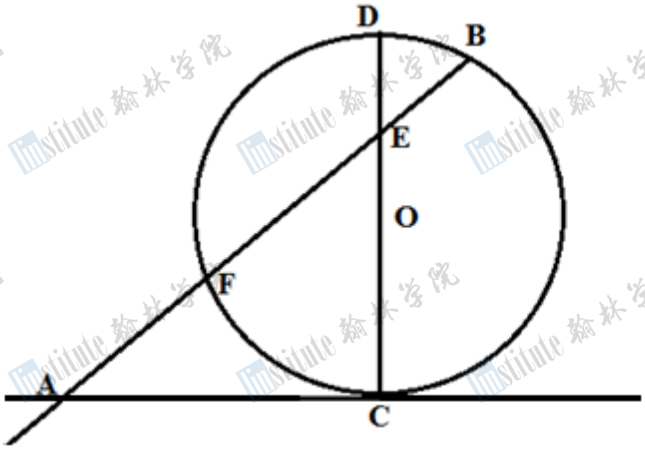
25. A city park has a monument with a tetrahedron shape as shown in figure.



Assume that the edges $\overline{BS}, \overline{CS}$, and $\overline{AS}$ of the tetrahedron are perpendicular to each other at the vertex S, and the areas of the faces (triangles) $\Delta ABS, \Delta BCS$, and ΔACS are 10, 11, and 12 square feet respectively. Find the area of triangle ΔABC.

A. 55 B. $\sqrt{365}$ C. 60 D. 66 E. $10\sqrt{3}$

26. The line $\overleftrightarrow{AC}$ is tangent to a circle with center O at C, and $\overline{CD}$ passes through center O. B is a point on the circle. $\overleftrightarrow{AB}$ intersects the circle and $\overline{CD}$ at F and E respectively. If $AF = 1, FE = 3$, and $EB = 2$, find the radius of the circle.

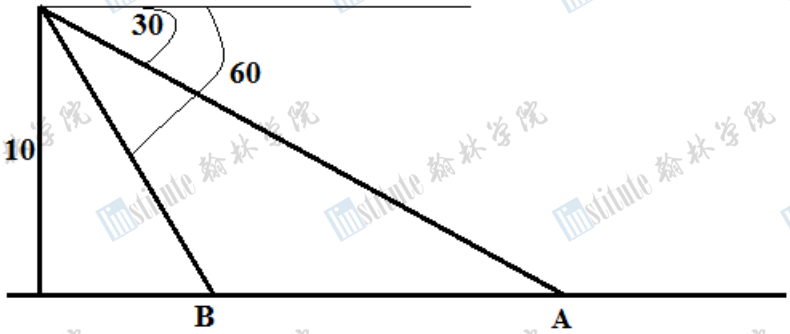


A. $\frac{4\sqrt{10}}{5}$ B. $\frac{3\sqrt{10}}{4}$ C. $\frac{3\sqrt{5}}{5}$ D. $\frac{4\sqrt{6}}{5}$ E. $\frac{3\sqrt{6}}{4}$

27. Find the area of the quadrilateral formed by the points of intersection of the curves $4x^2 + y^2 = 100$ and $9x^2 - y^2 = 108$.

A. 96 B. 84 C. 48 D. 24 E. 80

28. The pilot of an airplane flying at an altitude of 10*km* sees two towns, *A* and *B*, directly in view ahead. Suppose the pilot's line of view makes angles of 30^0 and 60^0 respectively with the horizontal. How far apart are the towns *A* and *B*?



A. $\frac{20\sqrt{3}}{3}$ B. 20 C. 10 D. $10\sqrt{3}$ E. None of these

29. Find the domain of the function

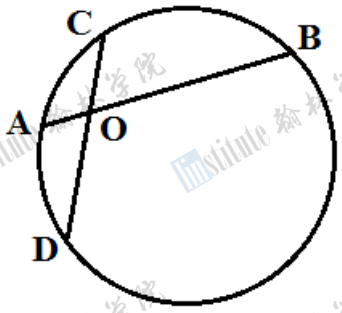
$$f(x) = \log_2\left(\sqrt{\frac{(1-x)(x+3)}{(x+1)(x-3)}}\right)$$

A. $(-1,1) \cup (1,3)$ B. $(-\infty,-3) \cup (-1,1) \cup (3,\infty)$ C. $[-3,-1) \cup (1,3]$
D. $(-3,-1) \cup (1,3)$ E. $(-3,-1] \cup [1,3)$

30. Let z_1 and z_2 be complex numbers such that $z_1 = 25 + i[(x-8)^2 + y^2]$ and $z_2 = x^2 + y^2 + 41i$, where *x* and *y* are real numbers.
Find the value of $x^2 - y^2$ such that $z_1 = z_2$.

A. –17 B. 9 C. 6 D. –13 E. None of these

31. Consider the circle shown in the figure. The chords $\overline{AB}$ and $\overline{DC}$ intersect at *O* such that $AO = 4cm$ and $DC = 16cm$. If $\overline{OC} = \overline{DO}$, find the length of $\overline{AB}$.

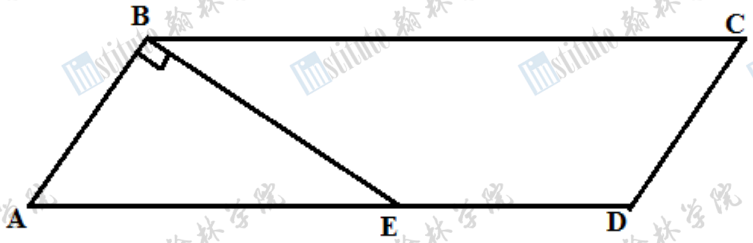


A. 16 *cm* B. 20 *cm* C. $4\sqrt{5}$ *cm*
D. $2\sqrt{38}$ *cm* E. 10 *cm*

32. Find the real solutions of the equation $2\ln(15+e^2)^{(2x^2+2)} = \ln(e^2+15)^{(3x^2+8x-11)}$.

A. $\{1,4\}$ B. $\{3,5\}$ C. $\{-5,-3\}$ D. $\{2,3\}$ E. $\{-3,-2\}$

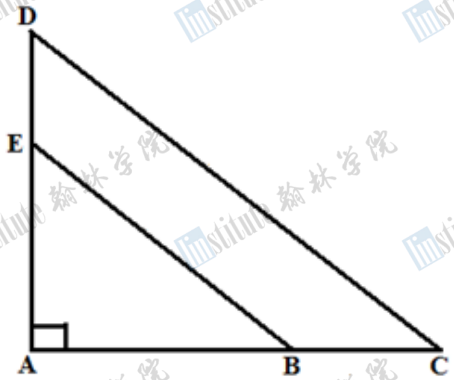
33. In parallelogram *ABCD*, $\overline{AB} = \overline{BE} = \overline{ED}$ and $\angle ABE = 90^0$. If the area of the parallelogram *ABCD* is $16cm^2$, how long is the side $\overline{AD}$?



A. $4(1+\sqrt{2})cm$ B. $4(2+2\sqrt{2})cm$ C. $4\sqrt{2+\sqrt{2}}cm$

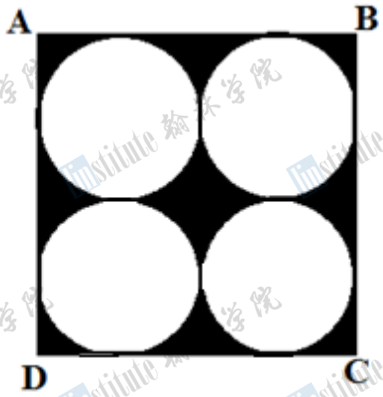
D. $10(1+\sqrt{2})cm$ E. $5(\sqrt[4]{2}+\sqrt{2})cm$

34. In the figure below $\overline{BE} \parallel \overline{CD}$, $AE = x$, $\angle EAB = 90°$, $BE = y$, $CD = 20$, $DE = 3$, $AB = 12$, $BC = 4$. Find the value of $x + y$.



A. 17 B. 32 C. 37 D. 28 E. None of these

35. In the figure below, the radius of each circle is 2 units. If a dart is thrown at random at the frame *ABCD*, find the probability that it will hit the shaded region.



A. $\frac{2-\pi}{2}$ B. 16π C. $\frac{\pi}{4}$

D. $\frac{4-\pi}{4}$ E. $\frac{8-\pi}{8}$

36. Suppose $C(x) = x^2 - 40x + 405$. Compute the minimum value of $C(x)$.

A. 5 B. 15 C. 10 D. 0 E. 405

37. Find the value of *k* such that $f(x) = x^3 - kx^2 + kx + 2$ has the factor $(x - 2)$.

A. 2 B. 10 C. 5 D. 15 E. 12

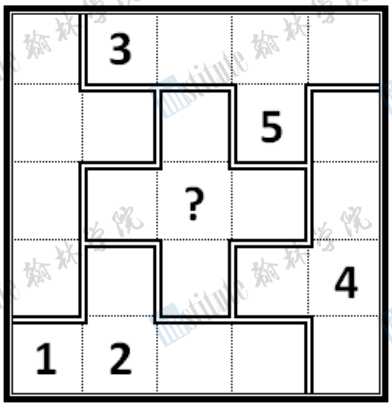
38. An urn contains 10 balls such that 4 of the balls are red and the rest are green. A second urn contains 16 red balls and *n* green balls. A single ball is drawn at random from each urn. The probability that the balls are of the same color is 0.44. Find the value of n – the number of green balls in the second urn.

A. 4 B. 8 C. 12 D. 16 E. 2

39. Consider the triangle *ABC* with vertices $A = (0,0)$, $B = (4,0)$, and $C = (0,8)$. Let triangle $A'B'C'$ be the image of triangle *ABC* after the transformation $T(x, y) = \left(\frac{1}{2}x, \frac{1}{4}y\right)$. Find the area of triangle $A'B'C'$.

A. 16 B. 8 C. 4 D. 1 E. 2

40. Enter the digits 1 through 5 into the cells in the grid on the right such that every row, every column, and every region contain each digit exactly once. Which digit belongs in the center?



A. 1 B. 2 C. 3 D. 4 E. 5