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Mathematics Contest

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NC STATE MATHEMATICS CONTEST
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PART I: 20 MULTIPLE CHOICE PROBLEMS

1. In a certain population the ratio of the number of women to the number of men is 11 to 10. If the average age of the women is 34 and the average age of the men is 32, find the average age of the population.

   (A) $32\frac{9}{10}$ (B) $32\frac{20}{21}$ (C) 33 **(D)** $33\frac{1}{21}$ (E) $33\frac{1}{10}$

   Solution: Answer: (D). Let w be the number of women and m be the number of men. Then $w = 11k$, $m = 10k$ for some positive integer k. The sum of the ages of the women is $34w$ and the sum of the ages of the men is $32m$. the average age of the population is

   $$\frac{34 \cdot 11k + 32 \cdot 10k}{11k + 10k} = \frac{694}{21} = 33\frac{1}{21}.$$

2. Let a, b, and c be positive integer numbers such that $a \log_{144} 3 + b \log_{144} 2 = c$. Find the value of $\frac{a+b}{c}$.

   (A) 6 (B) $\frac{11}{2}$ (C) 5 (D) 12 (E) None of the answers (A) through (D) is correct.

   Solution: Answer: (A). $a \log_{144} 3 + b \log_{144} 2 = c$ is equivalent to $3^{a-2c} = 2^{4c-b}$. Hence $a - 2c = 4c - b = 0$. Therefore, $\frac{a+b}{c} = 6$.

3. Let x_1 and x_2 be real roots of the function $f(x) = x^2 + bx + c$ such that $x_1 > 0$, $x_2 = 4x_1$, and a, b, c are real numbers. If $3 = 2(c - b)$, determine the value of x_1.

   (A) $\frac{1}{3}$ (B) $\frac{3}{5}$ (C) $\frac{1}{2}$ **(D)** $\frac{1}{4}$ (E) None of the answers (A) through (D) is correct.

   Solution: Answer: (D). $f(x) = (x - x_1)(x - x_2) = (x - x_1)(x - 4x_1) = x^2 - 5x_1x + 4x_1^2$. Hence, $b = -5x_1$, $c = 4x_1^2$. Since $3 = 2(c-b)$, we get $3 = 8x_1^2 + 10x_1$, i.e. $8x_1^2 + 10x_1 - 3 = 0$. This equation has one positive root, $x_1 = \frac{1}{4}$.

4. Three fair dice are tossed at random and all faces have the same probability of coming up. What is the probability that the three numbers turned up can be arranged to form an arithmetic progression with common difference one?

   (A) $\frac{1}{9}$ (B) $\frac{1}{27}$ (C) $\frac{1}{54}$ (D) $\frac{7}{36}$ (E) None of the answers (A) through (D) is correct.

   Solution: Answer: (A). The successful outcomes of the toss are the permutations of: (1,2,3), (2,3,4),(3,4,5), and (4,5,6). Hence, the probability that we are looking for is $\frac{6 \cdot 4}{6^3} = \frac{1}{9}$.

5. Let a, b, and c be the lengths of the sides of a triangle ABC. If $(a + b + c)(a + b - c) = 3ab$, then the measure of the angle opposite the side of length c is

   (A) 30° (B) 45° **(C) 60°** (D) 120° (E) 150°

Solution: Answer: (C). We get $a^2+b^2-ab=c^2$. Let γ be the angle opposite the side of length c. Then $a^2+b^2-2ab\cos\gamma=c^2$. Hence, $ab=2ab\cos\gamma$, i.e. $\cos\gamma=\frac{1}{2}$. Therefore, $\gamma=60°$.

6. Let x be a real number and y be a positive real number. If $|x-\log y|=x+\log y$, then

   (A) $x=0$ (B) $y=1$ (C) $x=0$ and $y=1$
   (D) $x(y-1)=0$ (E) None of the answers (A) through (D) is correct.

Solution: Answer: (D). If $x-\log y\ge 0$, then $x-\log y=x+\log y$, i.e. $\log y=0$. Hence, $y=1$. If $x-\log y<0$, then $-(x-\log y)=x+\log y$, i.e. $x=0$. Therefore, $x=0$ or $y=1$, which is equivalent to $x(y-1)=0$.

7. Let $a+bi$, $(b\neq 0)$ be a complex root of the equation $x^3+qx+r=0$ where a, b, q, and r are real numbers. Then q in terms of a and b is

   (A) a^2+b^2 (B) b^2-a^2 (C) $2a^2-b^2$ **(D)** b^2-3a^2 (E) b^2-2a^2

Solution: Answer: (D). If $x=a+bi$ is a root of the cubic equation, then $y=a-bi$ must be a root as well. The third root of the equation, denote it by z, must be a real number. Then $x+y+z=0$ and $xy+yz+xz=q$. From the first equation we get $z=-2a$. From the second equation we get $q=b^2-3a^2$.

8. Consider two real functions $f(x)=x^2+2bx+1$ and $g(x)=2a(x+b)$, where the constants a and b are real numbers. Let S be the set of all ordered pairs (a,b) for which the graphs of $y=f(x)$ and $y=g(x)$ do not intersect. The area of S is

   (A) 0 (B) 1 **(C)** π (D) infinite (E) None of the answers (A) through (D) is correct.

Solution: Answer: (C). If $f(x)=g(x)$, then $x^2+2(b-a)x+(1-2ab)=0$. This quadratic equation does not have real solutions if its discriminant is negative. Hence, the graphs of f and g do not intersect if $(2(b-a))^2-4(1-2ab)<0$. This implies that $a^2+b^2<1$. Thus S is the unit disc without the boundary, and its area is π.

9. The probability that an event A occurs is $\frac{3}{4}$ and the probability that event B occurs is $\frac{2}{3}$. Let p be the probability that both events A and B occur. The smallest interval necessarily containing p is

   (A) $\left[\frac{1}{2},\frac{2}{3}\right]$ (B) $\left[\frac{1}{12},\frac{1}{2}\right]$ (C) $\left[\frac{1}{12},\frac{2}{3}\right]$ (D) $\left[\frac{5}{12},\frac{1}{2}\right]$ **(E)** $\left[\frac{5}{12},\frac{2}{3}\right]$

Solution: Answer: (E). $p=P(A\cap B)=P(A)+P(B)-P(A\cup B)=\frac{3}{4}+\frac{2}{3}-P(A\cup B)$. Since $P(A\cup B)\le 1$ and $P(A\cup B)\ge\max\{P(A),P(B)\}=\frac{3}{4}$, we get $\frac{3}{4}+\frac{2}{3}-1\le p\le\frac{3}{4}+\frac{2}{3}-\frac{3}{4}$, i.e. $\frac{5}{12}\le p\le\frac{2}{3}$.

10. Evaluate the sum $\cos^2 1°+\cos^2 2°+\cos^2 3°+\cdots+\cos^2 88°+\cos^2 89°$.

   (A) $\frac{89}{2}$ (B) 45 (C) 44 (D) $\frac{181}{4}$ (E) None of the answers (A) through (D) is correct.

Solution: Answer: (A). Let $S = \cos^2 1° + \cos^2 2° + \cos^2 3° + \cdots + \cos^2 88° + \cos^2 89°$. Since $\cos^2\alpha = \sin^2(90° - \alpha)$ for $\alpha = 1°, 2°, \ldots, 89°$, we get $S = \sin^2 1° + \sin^2 2° + \sin^2 3° + \cdots + \sin^2 88° + \sin^2 89°$. Hence $2S = 89$, i.e. $S = \frac{89}{2}$.

11. Peter left town A at x minutes past 6:00 PM and reached town B at y minutes past 6:00 PM the same day. He noticed that at both the beginning and the end of the trip, the minute hand made the same angle of 110° with the hour hand on his watch. How many minutes did it take Peter to go from town A to town B?

(A) 38 (B) 39 **(C) 40** (D) 41 (E) None of the answers (A) through (D) is correct.

Solution: Answer: (C). It is clear that $0 < x < y < 60$. An angle of 110° corresponds to $\frac{55}{3}$ minutes. Hence.

$$\left(30 + \frac{x}{12}\right) - x = \frac{55}{3}, \quad y - \left(30 + \frac{y}{12}\right) = \frac{55}{3}$$

From these two equations we get $y - x = 40$.

12. Let a_n be a sequence defined by $a_n = \sum_{i=0}^{n} 2^i = 1 + 2 + 2^2 + \cdots + 2^n$ (n is a nonnegative integer). Find $\sum_{k=0}^{2015} a_k = a_0 + a_1 + a_2 + \cdots + a_{2015}$.

(A) $2^{2017} - 2018$ (B) $2^{2016} - 2017$ (C) $2016 \cdot 2^{2016}$ (D) $2^{2016} - 2015$
(E) None of the answers (A) through (D) is correct.

Solution: Answer: (A). First notice that $a_n = 2^{n+1} - 1$. Then $\sum_{k=0}^{2015} a_k = (2^1 - 1) + (2^2 - 1) + (2^3 - 1) + \cdots + (2^{2016} - 1) = (2^1 + 2^2 + 2^3 + \cdots + 2^{2016}) - 2015 = 2(1 + 2^1 + 2^2 + 2^{2015}) - 2016 = 2(2^{2016} - 1) - 2016 = 2^{2017} - 2018$

13. Let p be a polynomial function such that $p(x^2 + 1) = x^4 + 5x^2 + 3$ for all real numbers x. Determine $p(x^2 - 1)$ for all real x.

(A) $x^4 + 5x^2 + 1$ (B) $x^4 - 5x^2 + 1$ (C) $x^4 + x^2 + 3$ **(D) $x^4 + x^2 - 3$** (E) None of the answers (A) through (D) is correct.

Solution: Answer: (D). Notice $x^4 + 5x^2 + 3 = (x^4 + 2x^2 + 1) + (3x^2 + 3) - 1$. If we set $t = x^2 + 1$, we get $p(t) = t^2 + 3t - 1$. Then $p(x^2 - 1) = (x^2 - 1)^2 + 3(x^2 - 1) - 1 = x^4 + x^2 - 3$.

14. If the sum of the first 10 terms and the sum of the first 100 terms of an arithmetic progression are 100 and 10, respectively, then the sum of the first 110 terms is

(A) 0 **(B) -110** (C) -100 (D) -90 (E) None of the answers (A) through (D) is correct.

Solution: Answer: (B). Let a_1, d, and S_n, denote the first term, the difference, and the sum of the first n terms of the arithmetic sequence, respectively. Then $200 = 10(2a_1 + 9d)$ and $20 = 100(2a_1 + 99d)$. The sum of the first 110 terms is $S_{110} = \frac{110}{2}(2a_1 + 109d)$. From the first two equations we get $200 - 20 = 20a_1 + 90d - 200a_1 - 9900d$, i.e. $-2 = 2a_1 + 109d$. Then $S_{110} = -110$

15. Let p and q be prime numbers. Assume that the equation $x^2 - px + q = 0$ has distinct integer roots. Consider the following statements:

    I. The difference of the roots is odd.

    II. At least one root is prime.

    III. $p^2 - q$ is prime.

    IV. $p + q$ is prime.

    Which of the following is true?

    (A) I only (B) I and II only (C) III and IV only (D) II, III, and IV only **(E) All are true.**

    Solution: Answer: (E). Since the product of the roots is the prime number q, the roots must be 1 and q. Hence $p = 1 + q$. Since both p and q are prime, it follows that $p = 3$ and $q = 2$. Therefore all four statements are true.

16. A die is tossed. If the die lands on 1 or 2, then one coin is tossed. If the die lands on 3, then two coins are tossed. Otherwise, three coins are tossed. Given that the resulting coin tosses produced no "heads", what is the probability that the die landed on 1 or 2?

    (A) $\frac{13}{48}$ (B) $\frac{5}{13}$ (C) $\frac{17}{48}$ **(D) $\frac{8}{13}$** (E) None of the answers (A) through (D) is correct.

    Solution: Answer: (D). Consider the following events: A is the event that the die lands on 1 or 2; B is the event that the die lands on 3; C is the event that the die lands on 4 or 5 or 6. Then $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{6}$, $P(C) = \frac{1}{2}$. Let D be the event that no heads appear when the coin(s) are tossed. Then $P(D|A) = \frac{1}{2}$, $P(D|B) = \frac{1}{4}$, $P(D|C) = \frac{1}{8}$ and $P(D) = P(D|A)P(A)+P(D|B)P(B)+P(D|C)P(C) = \frac{13}{48}$. Finally, the probability that the die landed on 1 or 2 provided the resulting head tosses produced no "heads" is $P(A|D) = \frac{P(D|A)P(A)}{P(D)} = \frac{8}{13}$.

17. A triangle ABC is to be constructed given side a (opposite angle A), angle B, and h_c, the altitude from C. Then the number of non-congruent solutions must be

    (A) 0 (B) 1 (C) 2 (D) infinitely many **(E) None of the answers (A) through (D) is correct.**

    Solution: Answer: (E). The given parameters are not independent since $h_c = a \sin B$. If $h_c \neq a \sin B$, then there is no triangle with the given parameters. If $h_c = a \sin B$, then there are infinitely many triangles with the given parameters: we can choose the vertex A to be anywhere on the line BD (except B) where D is the foot of h_c. Therefore, there are zero or infinitely many triangles with the given parameters.

18. For the positive integer $a > 1$ by $\langle a \rangle$ we denote the greatest prime number not larger than a. How many positive integer solutions does the equation $\langle n+1 \rangle + \langle 2n+1 \rangle = \langle 5n+1 \rangle$ have?

    (A) 0 **(B) 1** (C) 2 (D) 4 (E) None of the answers (A) through (D) is correct.

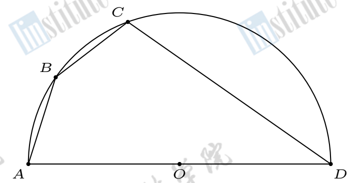
    Solution: Answer: (B) If $n \geq 2$, then $\langle n+1 \rangle$, $\langle 2n+1 \rangle$, and $\langle 5n+1 \rangle$ are all odd numbers so $\langle n+1 \rangle + \langle 2n+1 \rangle \neq \langle 5n+1 \rangle$. Hence, the equation does not have solutions if $n \geq 2$. If $n = 1$, then $\langle n+1 \rangle = 2$, $\langle 2n+1 \rangle = 3$, and $\langle 5n+1 \rangle = 5$. Hence, $n = 1$ is the unique solution of the equation.

19. Find the sum of all solutions of the equation $\sqrt[3]{x+20}-\sqrt[3]{x+1}=1$.

(A) 0 (B) 20 (C) 28 **(D) -21** (E) None of the answers (A) through (D) is correct.

Solution: Answer: (D) Let $a=\sqrt[3]{x+20}$ and $b=\sqrt[3]{x+1}$. Then $a-b=1$. Multiplying by a^2+ab+b^2 gives $a^3-b^3=a^2+ab+b^2$ or $a^2+ab+b^2=19$. Since $1=(a-b)^2=a^2-2ab+b^2$, we get $ab=6$. So $(x+20)(x+1)=a^3b^3=216$, i.e. $(x+28)(x-7)=0$. The solutions of the equation are $x=-28$ and $x=7$ and their sum is -21.

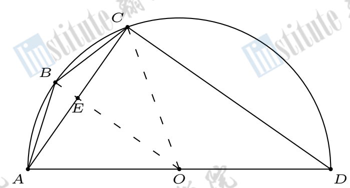
20. Quadrilateral $ABCD$ is inscribed in a circle with side AD as a diameter of length 4 cm. If sides AB and BC each have length 1 cm, find the length of the side CD.



(A) $\frac{7}{2}$ cm (B) 2 cm (C) 1 cm (D) $\frac{7}{4}$ cm (E) None of the answers (A) through (D) is correct.

Solution:
Answer: (A). The radius of $\overline{OB}$ is perpendicular to $\overline{AC}$ and bisects $\overline{AC}$ at E. $\overline{CD}$ is parallel to $\overline{EO}$ and $EO=\frac{1}{2}CD$. Since the triangles BEA and ABD are similar we have

$$\frac{BE}{1}=\frac{1}{AD}=\frac{1}{4}$$
$$EO=BO-BE=\frac{7}{4}$$



Hence $CD=2EO=\frac{7}{2}$.

PART II: 10 INTEGER ANSWER PROBLEMS

1. Andrew sells an item at \$10 less than the list price and receives 10% of his selling price as his commission. Tyler sells an item at \$20 less than the list price and receives 20% of his selling price as his commission. If they both get the same commission, what is the list selling price in dollars?

Solution: Answer: 30. Let x be the list price in dollars. Then $0.1(x-10)=0.2(x-20)$. Hence $x=30$.

2. Let $A=(2+1)(2^2+1)(2^4+1)(2^8+1)\cdots(2^{1024}+1)+1$. Find $\sqrt[1024]{A}$.

Solution: Answer: 4. $A=(2-1)(2+1)(2^2+1)(2^4+1)(2^8+1)\cdots(2^{1024}+1)=(2^2-1)(2^2+1)(2^4+1)(2^8+1)\cdots(2^{1024}+1)=(2^4-1)(2^4+1)(2^8+1)\cdots(2^{1024}+1)=\ldots=(2^{2048}-1)+1=2^{2048}$. Hence, $\sqrt[1024]{A}=4$.

3. Let a, b, c, and d be positive integers such that $a^5 = b^4$, $c^3 = d^2$ and $c - a = 19$. Determine the value of $d - b$.

> **Solution:** Answer: 757. Since 4 is relatively prime to 5 and 2 is relatively prime to 3, by the uniqueness of prime factorization, we get that there are positive integers x and y such that $a = x^4$, $b = x^5$, $c = y^2$, $d = y^3$. Then $(y - x^2)(y + x^2) = 19$. Since 19 is prime and $y - x^2 < y + x^2$, we get $y + x^2 = 19$ and $y - x^2 = 1$. Hence, $y = 10$, $x = 3$. Therefore, $d = 1000$, $b = 243$, and $d - b = 757$.

4. A set of consecutive integers beginning with 1 is written on a blackboard. One number is erased. The average (mean) of the remaining numbers is $35\frac{7}{17}$. What number was erased?

> **Solution:** Answer: 7. Let n be the last number on the board. The largest possible average (if 1 is erased) is $\frac{n+2}{2}$ and the smallest possible average (if n is erased) is $\frac{n}{2}$. Hence, $\frac{n}{2} \le 35\frac{7}{17} \le \frac{n+2}{2}$, i.e. $68\frac{14}{17} \le n \le 70\frac{14}{17}$. We get $n = 69$ or $n = 70$. Since $35\frac{7}{17}$ is the average of $(n-1)$ integers, the number $35\frac{7}{17}(n-1)$ must be integer. Hence, $n = 69$. Let a be the erased number. Then $\frac{\frac{69\cdot 70}{2} - a}{68} = 35\frac{7}{17}$. Therefore, $a = 7$.

5. Let a, b, and c be positive real numbers such that $a + b + c = 1$. Find the minimum value of

$$\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right)$$

> **Solution:** Answer: 8. Using $a + b + c = 1$ and the inequality between arithmetic and geometric mean, we get $\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right) = \frac{1-a}{a}\frac{1-b}{b}\frac{1-c}{c} = \frac{b+c}{a}\frac{a+c}{b}\frac{a+b}{c} \ge \frac{2\sqrt{bc}}{a}\frac{2\sqrt{ac}}{b}\frac{2\sqrt{ab}}{c} = 8$. Iquality holds if $a = b = c = \frac{1}{3}$. Hence, the minimum value is 8.

6. How many four-digit positive integers divisible by 7 have the property that, when the first and the last digit are interchanged, the result is a (not necessarily four-digit number) positive integer divisible by 7?

> **Solution:** Answer: 210. Suppose that $\overline{abcd} = 1000a + 100b + 10c + d$ is a four-digit number divisible by 7. If we interchange the first and the last digit we get the number $\overline{dbca} = 1000d + 100b + 10c + a$ which is also divisible by 7. Hence, their difference $999(a - d)$ is also divisible by 7. Since 999 is not divisible by 7, then $a - d$ must be divisible by 7, i.e. $a \equiv d \pmod 7$. Since $7|\overline{abcd}$, we get $1000a + 100b + 10c + d \equiv 0$ (mod 7, i.e. $-a + 10(10b + c) + d \equiv 0 \pmod 7$. Hence, $10b + c \equiv 0 \pmod 7$. There are 15 pairs b and c such that $10b + c \equiv 0 \pmod 7$, and 14 pairs a and d such that $a \equiv d \pmod 7$ $(a \ne 0)$ (there are 9 pairs such that $a = d$, three pairs such that $a = d + 7$, and two pairs such that $d = a + 7$). Therefore, the number we are looking for is $15 \cdot 14 = 210$.

7. Let ABC be a triangle, let D be a point on the side $\overline{AB}$, E be a point on the side $\overline{AC}$ such that $\overline{DE} \parallel \overline{BC}$ and $AD > BD$. Drop the perpendiculars from D and E to $\overline{BC}$, meeting $\overline{BC}$ at F and K, respectively. If $\frac{\text{Area}(ABC)}{\text{Area}(DEKF)} = \frac{32}{7}$, determine $\frac{AD}{DB}$.

Solution: Answer: 7. Let $\overline{AH}$ be the altitude from A to $\overline{BC}$ intersecting $\overline{DE}$ at L. From $\frac{\text{Area}(ABC)}{\text{Area}(DEKF)} = \frac{32}{7}$ we get $\frac{AH \cdot BC}{2} = \frac{32}{7} \cdot DE \cdot LH$, i.e. $\frac{AH}{LH} = \frac{64}{7} \cdot \frac{DE}{BC}$. Since the triangles ABC and ADE are similar, we get $\frac{AD}{AB} = \frac{DE}{BC}$. Since $\overline{DE} \parallel \overline{BC}$, we get $\frac{AB}{DB} = \frac{AH}{LH}$. Hence, $\frac{AB}{DB} = \frac{64}{7} \cdot \frac{AD}{AB}$. Then $\frac{AB^2}{DB^2} = \frac{64}{7} \cdot \frac{AD}{AB} \cdot \frac{AB}{DB} = \frac{64}{7} \cdot \frac{AD}{DB}$. Then $\left(\frac{AD+DB}{DB}\right)^2 = \frac{64}{7} \cdot \frac{AD}{DB}$, i.e. $\left(\frac{AD}{DB} + 1\right)^2 = \frac{64}{7} \cdot \frac{AD}{DB}$. We get the quadratic equation $\left(\frac{AD}{DB}\right)^2 - \frac{50}{7} \cdot \frac{AD}{DB} + 1 = 0$. The solutions are $\frac{AD}{DB} = \frac{1}{7}$ or $\frac{AD}{DB} = 7$. Since $AD > BD$, we get $\frac{AD}{BD} = 7$.

8. How many distinct ordered pairs, (x, y), of integer solutions does the equation

$$4x + y + 4\sqrt{xy} - 28\sqrt{x} - 14\sqrt{y} + 48 = 0$$

have?

Solution: Answer: 9. The given equation is equivalent to $(2\sqrt{x})^2 + 2(2\sqrt{x})(\sqrt{y}) + (\sqrt{y})^2 - 28\sqrt{x} - 14\sqrt{y} + 48 = 0$, and the last equation is equivalent to $(2\sqrt{x} + \sqrt{y} - 6)(2\sqrt{x} + \sqrt{y} - 8) = 0$. The equation $2\sqrt{x} + \sqrt{y} - 6 = 0$ is satisfied for $(0, 36), (1, 16), (4, 4), (9, 0)$, and the equation $2\sqrt{x} + \sqrt{y} - 8 = 0$ is satisfied for $(0, 64), (1, 36), (4, 16), (9, 4), (16, 0)$.

9. Find the largest value of the real parameter a for which the following system

$$\begin{cases} y = ax + 5 \\ (|x| + |y| - 2)^2 = 1 \end{cases}$$

has exactly three solutions.

Solution: Answer: 5. The second equation is satisfied if either $|x| + |y| = 3$ or $|x| + |y| = 1$. The graph of the first equation is a square with vertices $(3, 0)$, $(0, 3)$, $(-3, 0)$ and $(0, -3)$; the graph of the second equation is a square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$ (draw the graphs!). $y = ax + 5$ are lines through $(0, 5)$. There are only two of these lines that intersect the squares at exactly three points: one of them passes through $(-1, 0)$ and the other passes through $(1, 0)$. The largest possible value of the parameter a is 5.

10. Find the sum of all integer numbers x for which $\log_2(x^2 - 4x - 1)$ is also an integer number.

Solution: Answer: 4. Let n be an integer such that $\log_2(x^2 - 4x - 1) = n$. Then $x^2 - 4x - (1 + 2^n) = 0$. The solutions of this equation are $x = 2 - \sqrt{5 + 2^n}$ and $x = 2 + \sqrt{5 + 2^n}$. Since we are looking for integer x, then there is $k \in \mathbb{Z}$ such that $5 + 2^n = k^2$. It is easy to see that n must be positive integer. Since $5 + 2^n$ is odd, then k must be odd. Let $k = 2m - 1$ for some $m \in \mathbb{Z}$. Then $m(m-1) = 2^{n-2} + 1$. If $n = 1$, then $2^{n-2} + 1$ is not integer. If $n > 2$, then $2^{n-2} + 1$ is odd and $m(m-1)$ is even. Hence, $n = 2$ and $5 + 2^2 = 9$ and the solutions are $x = -1$ and $x = 5$. It is easy to check that for these two values of x, $\log_2(x^2 - 4x - 1)$ is indeed an integer. Therefore, theirs sum is 4.

The following problem, will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test.

TIE BREAKER PROBLEM

Find the sum of all real solutions of the equation $(x+1)(x^2+1)(x^3+1) = 30x^3$

Solution: Answer: 3. The equation is $x^6+x^5+x^4-28x^3+x^2+x+1=0$. Clearly $x=0$ is not a solution. Then the equation is equivalent to $x^3+x^2+x-28+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}=0$. Let $z=x+\frac{1}{x}$. Then $z^2=x^2+2+\frac{1}{x^2}$, i.e. $x^2+\frac{1}{x^2}=z^2-2$. We also have $z^3=x^3+3(x+\frac{1}{x})+\frac{1}{x^3}$, i.e. $x^3+\frac{1}{x^3}=z^3-3z$. Thus, $z^3+z^2-2z-30=0$. One real solution is $z=3$. Then the last equation is equivalent to $(z-3)(z^2+4z+10)=0$, and $z=3$ is the only real solution of this equation. Hence, $x+\frac{1}{x}=3$, and the real solutions of the equation are $\frac{3+\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$. Their sum is 3.