# The Thirty-seventh Annual State High School Mathematics Contest

## Thursday, April 23, 2015

Held on the Campus
of the North Carolina School
of Science and Mathematics
Durham, NC

Sponsored by
The North Carolina Council
of Teachers of Mathematics

# NC STATE MATHEMATICS CONTEST
# APRIL 2015

## PART I: 20 MULTIPLE CHOICE PROBLEMS

1. In a certain population the ratio of the number of women to the number of men is 11 to 10. If the average age of the women is 34 and the average age of the men is 32, find the average age of the population.

   (A) $32\frac{9}{10}$ (B) $32\frac{20}{21}$ (C) 33 (D) $33\frac{1}{21}$ (E) $33\frac{1}{10}$

2. Let $a$, $b$, and $c$ be positive integer numbers such that $a\log_{144}3 + b\log_{144}2 = c$. Find the value of $\frac{a+b}{c}$.

   (A) 6 (B) $\frac{11}{2}$ (C) 5 (D) 12 (E) None of the answers (A) through (D) is correct.

3. Let $x_1$ and $x_2$ be real roots of the function $f(x) = x^2 + bx + c$ such that $x_1 > 0$, $x_2 = 4x_1$, and $a$, $b$, $c$ are real numbers. If $3 = 2(c - b)$, determine the value of $x_1$.

   (A) $\frac{1}{3}$ (B) $\frac{3}{5}$ (C) $\frac{1}{2}$ (D) $\frac{1}{4}$ (E) None of the answers (A) through (D) is correct.

4. Three fair dice are tossed at random and all faces have the same probability of coming up. What is the probability that the three numbers turned up can be arranged to form an arithmetic progression with common difference one?

   (A) $\frac{1}{9}$ (B) $\frac{1}{27}$ (C) $\frac{1}{54}$ (D) $\frac{7}{36}$ (E) None of the answers (A) through (D) is correct.

5. Let $a$, $b$, and $c$ be the lengths of the sides of a triangle $ABC$. If $(a+b+c)(a+b-c) = 3ab$, then the measure of the angle opposite the side of length $c$ is

   (A) 30° (B) 45° (C) 60° (D) 120° (E) 150°

6. Let $x$ be a real number and $y$ be a positive real number. If $|x - \log y| = x + \log y$, then

   (A) $x = 0$ (B) $y = 1$ (C) $x = 0$ and $y = 1$
   (D) $x(y-1) = 0$ (E) None of the answers (A) through (D) is correct.

7. Let $a + bi$, $(b \neq 0)$ be a complex root of the equation $x^3 + qx + r = 0$ where $a$, $b$, $q$, and $r$ are real numbers. Then $q$ in terms of $a$ and $b$ is

   (A) $a^2 + b^2$ (B) $b^2 - a^2$ (C) $2a^2 - b^2$ (D) $b^2 - 3a^2$ (E) $b^2 - 2a^2$

8. Consider two real functions $f(x) = x^2 + 2bx + 1$ and $g(x) = 2a(x + b)$, where the constants $a$ and $b$ are real numbers. Let $S$ be the set of all ordered pairs $(a, b)$ for which the graphs of $y = f(x)$ and $y = g(x)$ do not intersect. The area of $S$ is

   (A) 0 (B) 1 (C) $\pi$ (D) infinite (E) None of the answers (A) through (D) is correct.

9. The probability that an event $A$ occurs is $\frac{3}{4}$ and the probability that event $B$ occurs is $\frac{2}{3}$. Let $p$ be the probability that both events $A$ and $B$ occur. The smallest interval necessarily containing $p$ is

(A) $\left[\frac{1}{2}, \frac{2}{3}\right]$ (B) $\left[\frac{1}{12}, \frac{1}{2}\right]$ (C) $\left[\frac{1}{12}, \frac{2}{3}\right]$ (D) $\left[\frac{5}{12}, \frac{1}{2}\right]$ (E) $\left[\frac{5}{12}, \frac{2}{3}\right]$

10. Evaluate the sum $\cos^2 1° + \cos^2 2° + \cos^2 3° + \cdots + \cos^2 88° + \cos^2 89°$.

(A) $\frac{89}{2}$ (B) 45 (C) 44 (D) $\frac{181}{4}$ (E) None of the answers (A) through (D) is correct.

11. Peter left town A at $x$ minutes past 6:00 PM and reached town B at $y$ minutes past 6:00 PM the same day. He noticed that at both the beginning and the end of the trip, the minute hand made the same angle of $110°$ with the hour hand on his watch. How many minutes did it take Peter to go from town A to town B?

(A) 38 (B) 39 (C) 40 (D) 41 (E) None of the answers (A) through (D) is correct.

12. Let $a_n$ be a sequence defined by $a_n = \sum_{i=0}^{n} 2^i = 1 + 2 + 2^2 + \cdots + 2^n$ ($n$ is a nonnegative integer). Find $\sum_{k=0}^{2015} a_k = a_0 + a_1 + a_2 + \cdots + a_{2015}$.

(A) $2^{2017} - 2018$ (B) $2^{2016} - 2017$ (C) $2016 \cdot 2^{2016}$ (D) $2^{2016} - 2015$
(E) None of the answers (A) through (D) is correct.

13. Let $p$ be a polynomial function such that $p(x^2+1) = x^4 + 5x^2 + 3$ for all real numbers $x$. Determine $p(x^2-1)$ for all real $x$.

(A) $x^4 + 5x^2 + 1$ (B) $x^4 - 5x^2 + 1$ (C) $x^4 + x^2 + 3$ (D) $x^4 + x^2 - 3$ (E) None of the answers (A) through (D) is correct.

14. If the sum of the first 10 terms and the sum of the first 100 terms of an arithmetic progression are 100 and 10, respectively, then the sum of the first 110 terms is

(A) 0 (B) -110 (C) -100 (D) -90 (E) None of the answers (A) through (D) is correct.

15. Let $p$ and $q$ be prime numbers. Assume that the equation $x^2 - px + q = 0$ has distinct integer roots. Consider the following statements:

I. The difference of the roots is odd.
II. At least one root is prime.
III. $p^2 - q$ is prime.
IV. $p + q$ is prime.

Which of the following is true?

(A) I only (B) I and II only (C) III and IV only (D) II, III, and IV only (E) All are true.

16. A die is tossed. If the die lands on 1 or 2, then one coin is tossed. If the die lands on 3, then two coins are tossed. Otherwise, three coins are tossed. Given that the resulting coin tosses produced no "heads", what is the probability that the die landed on 1 or 2?

(A) $\frac{13}{48}$ (B) $\frac{5}{13}$ (C) $\frac{17}{48}$ (D) $\frac{8}{13}$ (E) None of the answers (A) through (D) is correct.

17. A triangle $ABC$ is to be constructed given side $a$ (opposite angle $A$), angle $B$, and $h_c$, the altitude from $C$. Then the number of non-congruent solutions must be

(A) 0 (B) 1 (C) 2 (D) infinitely many (E) None of the answers (A) through (D) is correct.

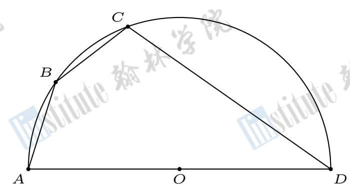
18. For the positive integer $a > 1$ by $\langle a \rangle$ we denote the greatest prime number not larger than $a$. How many positive integer solutions does the equation $\langle n+1 \rangle + \langle 2n+1 \rangle = \langle 5n+1 \rangle$ have?

(A) 0 (B) 1 (C) 2 (D) 4 (E) None of the answers (A) through (D) is correct.

19. Find the sum of all solutions of the equation $\sqrt[3]{x+20} - \sqrt[3]{x+1} = 1$.

(A) 0 (B) 20 (C) 28 (D) $-21$ (E) None of the answers (A) through (D) is correct.

20. Quadrilateral $ABCD$ is inscribed in a circle with side $AD$ as a diameter of length 4 cm. If sides $AB$ and $BC$ each have length 1 cm, find the length of the side $CD$.




(A) $\frac{7}{2}$ cm (B) 2 cm (C) 1 cm (D) $\frac{7}{4}$ cm (E) None of the answers (A) through (D) is correct.

# PART II: 10 INTEGER ANSWER PROBLEMS

1. Andrew sells an item at \$10 less than the list price and receives 10% of his selling price as his commission. Tyler sells an item at \$20 less than the list price and receives 20% of his selling price as his commission. If they both get the same commission, what is the list selling price in dollars?

2. Let $A = (2+1)(2^2+1)(2^4+1)(2^8+1)\cdots(2^{1024}+1)+1$. Find $\sqrt[1024]{A}$.

3. Let $a$, $b$, $c$, and $d$ be positive integers such that $a^5 = b^4$, $c^3 = d^2$ and $c - a = 19$. Determine the value of $d - b$.

4. A set of consecutive integers beginning with 1 is written on a blackboard. One number is erased. The average (mean) of the remaining numbers is $35\frac{7}{17}$. What number was erased?

5. Let $a$, $b$, and $c$ be positive real numbers such that $a + b + c = 1$. Find the minimum value of

$$\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)$$

6. How many four-digit positive integers divisible by 7 have the property that, when the first and the last digit are interchanged, the result is a (not necessarily four-digit number) positive integer divisible by 7?

7. Let $ABC$ be a triangle, let $D$ be a point on the side $\overline{AB}$, $E$ be a point on the side $\overline{AC}$ such that $\overline{DE} \parallel \overline{BC}$ and $AD > BD$. Drop the perpendiculars from $D$ and $E$ to $\overline{BC}$, meeting $\overline{BC}$ at $F$ and $K$, respectively. If $\frac{\text{Area}(ABC)}{\text{Area}(DEKF)} = \frac{32}{7}$, determine $\frac{AD}{DB}$.

8. How many distinct ordered pairs, $(x, y)$, of integer solutions does the equation

$$4x + y + 4\sqrt{xy} - 28\sqrt{x} - 14\sqrt{y} + 48 = 0$$

have?

9. Find the largest value of the real parameter $a$ for which the following system

$$\begin{cases} y = ax + 5 \\ (|x| + |y| - 2)^2 = 1 \end{cases}$$

has exactly three solutions.

10. Find the sum of all integer numbers $x$ for which $\log_2(x^2 - 4x - 1)$ is also an integer number.

The following problem, will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test.

# TIE BREAKER PROBLEM

Find the sum of all real solutions of the equation $(x+1)(x^2+1)(x^3+1) = 30x^3$.