The Thirty-seventh Annual State High School Mathematics Contest

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Thursday, April 23, 2015

Held on the Campus of the North Carolina School of Science and Mathematics Durham, NC

Sponsored by The North Carolina Council of Teachers of Mathematics

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NC STATE MATHEMATICS CONTEST APRIL 2015

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而时间的新林塔梯 PART I: 20 MULTIPLE CHOICE PROBLEMS

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1. In a certain population the ratio of the number of women to the number of men is 11 to 10. If the average age of the women is 34 and the average age of the men is 32, find the average age of the population.

(C) 33 (D) $33\frac{1}{21}$ (E) $33\frac{1}{10}$ (A) $32\frac{9}{10}$ (B) $32\frac{20}{21}$

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- 2. Let a, b, and c be positive integer numbers such that $a \log_{144} 3 + b \log_{144} 2 = c$. Find the value of $\frac{a+b}{c}$.
 - (A) 6 (B) $\frac{11}{2}$ (C) 5_{12} (D) 12 (E) None of the answers (A) through (D) is correct.
- 3. Let x_1 and x_2 be real roots of the function $f(x) = x^2 + bx + c$ such that $x_1 > 0$, $x_2 = 4x_1$, and a, b, care real numbers. If 3 = 2(c - b), determine the value of x_1 .
 - (B) $\frac{3}{5}$ (C) $\frac{1}{2}$ (D) $\frac{1}{4}$ (E) None of the answers (A) through (D) is correct. (A) $\frac{1}{3}$
- 4. Three fair dice are tossed at random and all faces have the same probability of coming up. What is the probability that the three numbers turned up can be arranged to form an arithmetic progression with common difference one?
 - (B) $\frac{1}{27}$ (C) $\frac{1}{54}$ (A) $\frac{1}{9}$ (D) $\frac{7}{36}$ (E) None of the answers (A) through (D) is correct.

5. Let a, b, and c be the lengths of the sides of a triangle ABC. If (a + b + c)(a + b - c) = 3ab, then the measure of the angle opposite the side of length c is (A) 20° (B) 45° (C) 60° (D) 120° (E) 150° ute Wark

- (B) 45° (C) 60° (D) 120° (A) 30° (E) 150°
- 6. Let x be a real number and y be a positive real number. If $|x \log y| = x + \log y$, then
- (A) x = 0 (B) y = 1 (C) x = 0 and y = 1(D) x(y-1) = 0 (E) None of the answers (A) through (D) is correct.
- 7. Let a + bi, $(b \neq 0)$ be a complex root of the equation $x^3 + qx + r = 0$ where a, b, q, and r are real numbers. Then q in terms of a and b is
 - (C) $2a^2 b^2$ (D) $b^2 - 3a^2$ (E) $b^2 - 2a^2$ (A) $a^2 + b^2$ (B) $b^2 - a^2$
- 8. Consider two real functions $f(x) = x^2 + 2bx + 1$ and g(x) = 2a(x+b), where the constants a and b are real numbers. Let S be the set of all ordered pairs (a, b) for which the graphs of y = f(x) and y = g(x)do not intersect. The area of S is

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(B) 1(D) infinite (E) None of the answers (A) through (D) is correct. $(\mathbf{A}) \mathbf{0}$ (C) π Withte the the 's the Astitute the tet is the Astitute the the 'S the Astitute # # '% ! withte the the the stitute # # 13 PR

9. The probability that an event A occurs is $\frac{3}{4}$ and the probability that event B occurs is $\frac{2}{3}$. Let p be the tute the the probability that both events A and B occur. The smallest interval necessarily containing p is

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(C) $\begin{bmatrix} \frac{1}{12}, \frac{2}{3} \end{bmatrix}$ (D) $\begin{bmatrix} \frac{5}{12}, \frac{1}{2} \end{bmatrix}$ (E) $\begin{bmatrix} \frac{5}{12}, \frac{2}{3} \end{bmatrix}$ (A) $\left[\frac{1}{2}, \frac{2}{3}\right]$ (B) $\left[\frac{1}{12}, \frac{1}{2}\right]$

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- 10. Evaluate the sum $\cos^2 1^\circ + \cos^2 2^\circ + \cos^2 3^\circ + \dots + \cos^2 88^\circ + \cos^2 89^\circ$.
 - (D) $\frac{181}{4}$ $(A) \frac{\delta \varepsilon}{2}$ (E) None of the answers (A) through (D) is correct. (B) 45 (C) 44
- 11. Peter left town A at x minutes past 6:00 PM and reached town B at y minutes past 6:00 PM the same day. He noticed that at both the beginning and the end of the trip, the minute hand made the same angle of 110° with the hour hand on his watch. How many minutes did it take Peter to go from town A to town B?
 - (E) None of the answers (A) through (D) is correct. (A) 38 (B) 39 (D) 41 (C) 40

stitute ## 12. Let a_n be a sequence defined by $a_n = \sum_{i=0}^n 2^i = 1 + 2 + 2^2 + \dots + 2^n$ (*n* is a nonnegative integer). Find 2015

- $\sum_{k=0}^{k} a_k = a_0 + a_1 + a_2 + \dots + a_{2015}.$ $\begin{array}{c} \underset{k=0}{\overset{(A)}{(A)}} 2^{2017} - 2018 \quad (B) \quad 2^{2016} - 2017 \quad (C) \quad 2016 \cdot 2^{2016} \quad (D) \quad 2^{2016} - 2015 \\ \end{array}$ (E) None of the answers (A) through (D) is correct.
- 13. Let p be a polynomial function such that $p(x^2 + 1) = x^4 + 5x^2 + 3$ for all real numbers x. Determine $p(x^2 - 1)$ for all real x.
 - (C) $x^4 + x^2 + 3$ (D) $x^4 + x^2 3$ (E) None of the answers (A) $x^4 + 5x^2 + 1$ (B) $x^4 - 5x^2 + 1$ (A) through (D) is correct.
- 14. If the sum of the first 10 terms and the sum of the first 100 terms of an arithmetic progression are 100 and 10, respectively, then the sum of the first 110 terms is
 - (A) $0 \circ (B) -110$ (C) -100 (D) -90 (E) None of the answers (A) through (D) is correct.
- 15. Let p and q be prime numbers. Assume that the equation $x^2 px + q = 0$ has distinct integer roots. Consider the following statements:
 - I. The difference of the roots is odd. 加斯林·洛梯
 - II. At least one root is prime.
- III. $p^2 q$ is prime.

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IV. p + q is prime.

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Which of the following is true?

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(A) I only (B) I and II only (C) III and IV only (D) II, III, and IV only (E) All are true. Withte the the 's the Withthe the the the PR Withit the the the the withte the the the Astitute ## # '\$ 1%

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16. A die is tossed. If the die lands on 1 or 2, then one coin is tossed. If the die lands on 3, then two coins are tossed. Otherwise, three coins are tossed. Given that the resulting coin tosses produced no "heads", what is the probability that the die landed on 1 or 2?

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(A) $\frac{13}{48}$ (B) $\frac{5}{13}$ (C) $\frac{17}{48}$ (D) $\frac{8}{13}$ (E) None of the answers (A) through (D) is correct.

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- 17. A triangle ABC is to be constructed given side a (opposite angle A), angle B, and h_c , the altitude from 1/2 1/2 C. Then the number of non-congruent solutions must be
 - (A) 0 (C) 2(D) infinitely many (E) None of the answers (A) through (D) is correct. (B) 1
- 18. For the positive integer a > 1 by $\langle a \rangle$ we denote the greatest prime number not larger than a. How many positive integer solutions does the equation $\langle n+1 \rangle + \langle 2n+1 \rangle = \langle 5n+1 \rangle$ have?

(C) 2 (D) 4 (E) None of the answers (A) through (D) is correct. (A) 0 (B) 1

19. Find the sum of all solutions of the equation $\sqrt[3]{x+20} - \sqrt[3]{x+1} = 1$. ute the th (B) 20 (C) 28 (D) -21 (E) None of the answers (A) through (D) is correct. (A) 0

20. Quadrilateral ABCD is inscribed in a circle with side AD as a diameter of length 4 cm. If sides AB and BC each have length 1 cm, find the length of the side CD.

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(B) 2 cm (C) 1 cm (D) $\frac{7}{4}$ cm (E) None of the answers (A) through (D) is correct. (A) $\frac{1}{2}$ cm Autitute # ****

PART II: 10 INTEGER ANSWER PROBLEMS

- 1. Andrew sells an item at \$10 less than the list price and receives 10% of his selling price as his commission. Tyler sells an item at \$20 less than the list price and receives 20% of his selling price as his commission. If they both get the same commission, what is the list selling price in dollars? 资本
- 2. Let $A = (2+1)(2^2+1)(2^4+1)(2^8+1)\cdots(2^{1024}+1)+1$. Find $\sqrt[1024]{A}$.
- 3. Let a, b, c, and d be positive integers such that $a^5 = b^4$, $c^3 = d^2$ and c a = 19. Determine the value of d-b.
- 4. A set of consecutive integers beginning with 1 is written on a blackboard. One number is erased. The average (mean) of the remaining numbers is $35\frac{7}{17}$. What number was erased?

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5. Let a, b, and c be positive real numbers such that a + b + c = 1. Find the minimum value of Withte the the the pre Institute # # '3 stitute # *** $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)$

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6. How many four-digit positive integers divisible by 7 have the property that, when the first and the last. digit are interchanged, the result is a (not necessarily four-digit number) positive integer divisible by 7?

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institute 34 7. Let ABC be a triangle, let D be a point on the side \overline{AB} , E be a point on the side \overline{AC} such that $\overline{DE} \parallel \overline{BC}$ and AD > BD. Drop the perpendiculars from D and E to \overline{BC} , meeting \overline{BC} at F and K, respectively. $\frac{\text{Area}(ABC)}{\text{Area}(DEKF)} = \frac{32}{7}, \text{ determine } \frac{AD}{DB}.$ If 而如此他称林塔梯

8. How many distinct ordered pairs, (x, y), of integer solutions does the equation $4x + y + 4\sqrt{xy} - 28\sqrt{x} - 14\sqrt{y} + 48 = 0$

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have?

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has exactly three solutions.

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Withte Star H 'S PR ·2 4 ·k th · 10 810 1/2 th 80 10. Find the sum of all integer numbers x for which $\log_2(x^2 - 4x - 1)$ is also an integer number.

The following problem, will be used only as part of a tie-breaking procedure. Do not work on it until mutute # # 3 PC you have completed the rest of the test. 10.1111111 新林 举 muitute # # B

TIE BREAKER PROBLEM

