North Carolina Mathematics Level I Contest

May 1, 2014

1. Compute the sum of the coefficients of the polynomial $P(x)$, given that
$$P(2x+1)+P(1+x^2)=2014+x+x^2+x^3+\cdots+x^{2014}.$$
(A) 2014 (B) 2013 (C) 1 (D) 1007 (E) None of these .

2. Simplify the expression $\left[\frac{1}{\sqrt{18}+\sqrt{2}}\right]^{100}$.
(A) $\frac{1}{250}$ (B) $\left(\sqrt{20}\right)^{100}$ (C) 50 (D) 2^{-250} (E) $\left(\frac{1}{\sqrt{20}}\right)^{100}$

3. Find all real solutions of the equation $16^{x^2+1}=32^{x^2}$.
(A) $\{\sqrt{2},-\sqrt{2}\}$ (B) $\{2,-2\}$ (C) $\{\sqrt{3},-\sqrt{3}\}$ (D) $\{3,-3\}$ (E) None of these

4. Let a, b, c, and d be the four real solutions of the equation $(x^2-8x-1)^2-64=0$. Compute the value of $a+b+c+d$.
(A) 16 (B) 64 (C) 32 (D) 8 (E) 128

5. How many integer solutions does the following inequality have? $y^2-4y+3\le 0$
(A) 2 (B) 3 (C) 4 (D) 5 (E) infinitely many

6. Consider a right triangle whose sides have length x, y and z, where $y-x=9$ and $z-x=1$, Find the perimeter of the triangle .
(A) 60 (B) 64 (C) 70 (D) 81 (E) 100

7. Let a, b, and c be nonzero real numbers such that $a+b+c=0$. Compute the value of
$$a\left(\frac{1}{b}+\frac{1}{c}\right)+b\left(\frac{1}{a}+\frac{1}{c}\right)+c\left(\frac{1}{a}+\frac{1}{b}\right).$$
(A) 0 (B) 1 (C) -1 (D) -3 (E) 3

8. Let a be a real number. Given that $a+\frac{1}{a}=3$, find the value of $a^2+\frac{1}{a^2}$.
(A) 1 (B) 7 (C) 9 (D) 5 (E) 3

9. Let m be the solution of the equation $\frac{x-3}{3} - \frac{x+3}{6} = 3 - \frac{x-1}{2}$. Compute the value of $2m^2 - m$.

(A) 153 (B) 5.5 (C) 15 (D) 105 (E) 28

10. Find the value of the expression $\sqrt{21-\sqrt{320}} + \sqrt{9-\sqrt{80}}$.

(A) 2 (B) 3 (C) 4 (D) 5 (E) 1

11. Find the remainder if $1! + 2! + 3! + \cdots + 2013! + 2014!$ is divided by 15.

(A) 10 (B) 4 (C) 3 (D) 14 (E) 1

12. Eight people compete in a downhill ski race. In how many different orders can the skiers finish the race if there are no ties?

(A) 8 (B) 7 (C) 8! (D) 11 (E) None of these

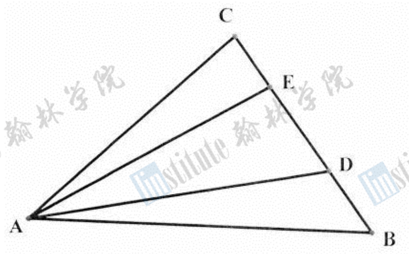
13. Consider three nonzero real numbers a, b, c, and d, such that: $\frac{|abcd|}{abcd} = 1$. Which of the following numbers cannot be the value of $\frac{|a|}{a} + \frac{|b|}{b} + \frac{|c|}{c}$?

(A) -3 (B) -1 (C) 1 (D) 3 (E) None of these

14. If 4 and 6 are solutions of equation $ax^2 + bx + c = 0$, find the solutions of equation $ax^2 - bx - c = 0$.

(A) $\{-2, 12\}$ (B) $\{-4, -6\}$ (C) $\{-12, 2\}$ (D) $\{-12, -2\}$ (E) $\{-4, 6\}$

15. Consider the triangle ABC with points D, E on side BC and D is between B and E. Suppose that the area of triangle ABD is a and the area of triangle AEC is b with $a > b$. If $BE = x$ and $DC = y$, Find the value of DE.



(A) $\frac{bx-ay}{b-a}$ (B) $\frac{bx+ay}{b+a}$ (C) $\frac{ax-by}{a-b}$ (D) $\frac{ax+by}{a+b}$ (E) $\frac{ax-by}{a+b}$

16. Let a, b, and c be three distinct integers such that $a + b + c = 10$. Find the integer solution of the equation: $(x - a)(x - b)(x - c) = -5$.

(A) 3 (B) 5 (C) 7 (D) 9 (E) None of these

17. Suppose $P(A \cup B) = 0.7$ and $P(A \cup \bar{B}) = 0.9$, where $P(\bar{B})$ is the probability that event B does not occur. Then $P(A)$ is

(A) 0.2 (B) 0.3 (C) 0.4 (D) 0.6 (E) 0.8

18. The arithmetic mean of the list of numbers $3, 8, 2, p, q, 3$ is 4. If p and q are integers and $p \neq q$, what is the median of the list?

(A) 2 (B) 2.5 (C) 3 (D) 3.5 (E) 4

19. A Law firm has 15 lawyers that will be randomly assigned to 15 new offices, 2 of which are corner offices. If everyone has an equal chance of getting a corner office, what is the probability that both Tony and Harry will get corner offices?

(A) $\frac{1}{15}$ (B) $\frac{2}{15}$ (C) $\frac{1}{105}$ (D) $\frac{1}{225}$ (E) $\frac{2}{225}$

20. Consider the contingency table depicting 500 individuals classified into the sub-categories $A = (A_1, A_2, A_3)$ and $B = (B_1, B_2, B_3)$ as shown below.

	B_1	B_2	B_3
A_1	20	30	20
A_2	60	140	60
A_3	20	80	70

If a person is selected at random from the group of 500 individuals, compute the probability that the person belongs to A_1 or B_1.

(A) 0.39 (B) 0.09 (C) 0.90 (D) 0.03 (E) 0.30

21. A projectile is fired vertically from the top of a tower which is 4 feet above the ground. The height $h(t)$ (in feet) of the projectile above the ground at time t seconds is given by the equation

$$h(t) = -16t^2 + 64t + 4.$$

Which of the following statements is (are) valid?

I. The projectile cannot attain a maximum height of 80 feet above the ground.

II. The time at which the projectile attains its maximum height is $t = 2$ seconds.

III. The projectile hits the ground at $t = 4$ seconds.

(A) I, II, and III (B) I and II only (C) II and III only (D) I and III only

(E) None of these

22. Find the values of y that solve the equation $ab(y^2+1) = y(a^2+b^2)$.

(A) $\left\{\frac{b}{a}, \frac{1}{b}\right\}$ (B) $\left\{\frac{a^2}{2b}, \frac{b^2}{2a}\right\}$ (C) $\left\{\frac{a}{b}, \frac{b}{a}\right\}$ (D) $\left\{\frac{a}{b}, b\right\}$ (E) $\left\{\frac{b}{a}, a\right\}$

23. Find the area of the region enclosed by the graphs of functions

$$f(x) = |x| + |x-4| \quad \text{and} \quad g(x) = 8.$$

(A) 12 (B) 48 (C) 24 (D) 32 (E) None of these

24. Suppose that x and y satisfy the following system of equations $\begin{cases} x+y=10 \\ x-y=6. \end{cases}$
Compute the value of x^2+y^2.

(A) 113 (B) 52 (C) 86 (D) 58 (E) 68

25. Function $g(x)$ is defined as follows: $g(x) = \begin{cases} -1, & if \quad x<0; \\ 0, & if \quad x=0; \\ 1, & if \quad x>0. \end{cases}$
Let $f(x) = g(x+1) - g(x-1)$. Compute $f\left(\frac{1}{2}\right) + f\left(-\frac{1}{2}\right)$.

(A) 0 (B) 2 (C) 4 (D) 1 (E) None of these

26. Find the distance between the points of intersection of the graphs of the following curves

$$9x^2+16y^2=144 \quad \text{and} \quad x^2-y^2=16.$$

(A) 10 (B) 9 (C) 8 (D) $2\sqrt{2}$ (E) 16

27. Let $f(x) = \frac{x-3}{x+5}$. Which of the following statements is (are) valid?
I. $f(x) \le 0$ on $(-5, 3]$.
II. $f(x) \ge 0$ on $(-\infty, -5) \cup [3, \infty)$.
III. $f(x) = 0$ on $\{3, -5\}$.
IV. $f(x) > 0$ on $(-5, 3)$.

(A) I only (B) I and II only (C) I, II, and III only (D) II and III only (E) None of these

28. Suppose that $\frac{-4x^3+3x^2-5}{x-2} = ax^2+bx+c+\frac{d}{x-2}$. Evaluate $3a-2b+c-d$.

(A) 25 (B) 13 (C) 0 (D) 33 (E) None of these

29. Let $\{a\}$ denote the decimal part of the real number a. For example $\{8.32\} = 0.32$. Let
$$S = \{\sqrt{1}\} + \{\sqrt{2}\} + \{\sqrt{3}\} + \cdots + \{\sqrt{100}\}.$$
Which of the following is (are) true?

(A) $S < 50$ (B) $50 < S < 60$ (C) $60 < S < 70$ (D) $70 < S < 80$ (E) $80 < S < 90$

30. Consider three non-zero real numbers a, b, and c such that $(a+b+c)c < 0$. Which of the following statements is always true?

(A) $b^2 < 4ac$ (B) $b^2 = 4ac$ (C) $b^2 > 4ac$ (D) $a^2 > 4ac$ (E) $c^2 = 4ab$

31. A person begins a dieting program that is designed to reduce his weight at least x lbs per week. At the beginning of the program, the person weighs M lbs. Let t be the maximum number of weeks that it will take for the person's weight to reach or fall below his goal of N lbs. Which of the following inequalities is true?

(A) $t > \frac{M-N}{x}$ (B) $t \leq \frac{M-N}{x}$ (C) $t \leq Mx - N$ (D) $t \leq \frac{Nx}{M}$ (E) $t \leq Nx - M$

32. Let a and b be real numbers that satisfy the equations $\begin{cases} a^2 + b^2 = 7 \\ a^2 + 2b^2 = 10 \end{cases}$.
Then the minimum value of $a + b$ is

(A) -1 (B) $-2-\sqrt{3}$ (C) 1 (D) $2+\sqrt{3}$ (E) -5

33. Three distinct rational numbers can be expressed in the form of 1, $a+b$, and a. They can also be expressed in the form of 0, $\frac{a}{b}$, and b. Find $a^{2014} + b^{2013}$.

(A) 0 (B) 1 (C) 2 (D) -1 (E) None of these

34. In the mythical galaxy of Jamais, the temperature of a cooling object is modeled by the equation
$$T(t) = T_A + (T_0 - T_A)10^{-kt}, \text{ where}$$
$T(t)$: the temperature of the object at time t
T_A: the ambient (environmental) temperature
T_0: the initial temperature of the object
k: the cooling constant
An object is heated to a temperature of 100100^0C in a furnace. Then the object is removed from the furnace and immediately placed in a room whose temperature is 100^0C. The temperature of the object at 9:00 am Monday is 10100^0C. Later on, at 7:00 pm on the same day, the temperature of the object is measured as 1100^0C. When was the object placed in the room?

(A) 3:00 am Sunday (B) 3:00 pm Sunday (C) 4:00 am Sunday (D) 4:00 pm Sunday
(E) 11:00 pm Sunday

35. In triangle ABC, $AB = AC = 2$. Which of the following can NOT be the area of triangle ABC?

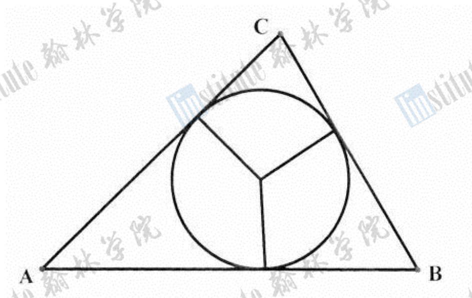
(A) 0.5 (B) 1.0 (C) 2.0 (D) 2.5 (E) 1.5

36. Let a, b, c, and d be four rational numbers and satisfy the following relation

$$\frac{1}{a-1997} = \frac{1}{b+1998} = \frac{1}{c-1999} = \frac{1}{d+2000}.$$ Which of the following is true?

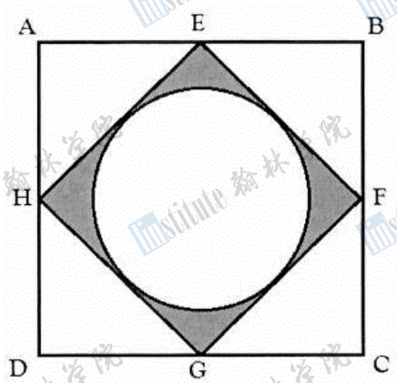
(A) $a > c > b > d$ (B) $b > d > a > c$ (C) $c > a > b > d$ (D) $d > b > a > c$
(E) $a > b > c > d$

37. Consider triangle ABC with an inscribed circle whose radius is r. If $AB = c$, $BC = a$, and $AC = b$, find the area of triangle ABC.



(A) $\frac{1}{2}r(a+b+c)$ (B) $\frac{3}{2}r(a+b+c)$ (C) $\frac{1}{2}abc$ (D) $r(a+b+c)$ (E) $\frac{1}{2}a(a+b+c)$

38. The square $ABCD$ has side of length 16 inches. A circle is inscribed in the quadrilateral $EFGH$ as shown below, where E, F, G, and H are midpoints of the sides of square $ABCD$. Find the area of the shaded region.



(A) $8\ in^2$ (B) $(8-\pi)\ in^2$ (C) $(128-32\pi)\ in^2$ (D) $(196-64\pi)\ in^2$ (E) None of these

39. Let p be a prime number different from 2. Consider the number
$N(p) = 2013p^2 + 2014p + 2015$. Which of the following statements is (are) valid?

I. N is divisible by 2.
II. N is an odd number.
III. N is a prime number for $p > 2$.

(A) III only (B) I only (C) II only (D) I and II only (E) II and III only

40. A box contains 10 balls, numbered from 1 to 10. If three balls are selected at random and with replacement from the box, what is the probability that the sum of the three numbers on the balls selected from the box will be even?

(A) $\frac{1}{4}$ (B) $\frac{3}{4}$ (C) $\frac{1}{2}$ (D) $\frac{3}{8}$ (E) $\frac{5}{8}$