# The Thirty-fifth Annual State High School Mathematics Contest

## Thursday, April 24, 2014

Held on the Campus
of the North Carolina School
of Science and Mathematics
Durham, NC

Sponsored by
The North Carolina Council
of Teachers of Mathematics

# NC STATE MATHEMATICS CONTEST
# APRIL 2014

## PART I: 20 MULTIPLE CHOICE PROBLEMS

1. For all positive real numbers $x$, $y$, $z$, the product

$$(x+y+z)^{-1}(x^{-1}+y^{-1}+z^{-1})(xy+yz+xz)^{-1}((xy)^{-1}+(yz)^{-1}+(xz)^{-1})$$

is equal to

(A) $\frac{1}{xyz}$ **(B) $x^{-2}y^{-2}z^{-2}$** (C) $(x+y+z)^{-1}$ (D) $x^{-2}+y^{-2}+z^{-2}$
(E) None of the answers (A) through (D) is correct.

**Solution:** (B)

$$(x+y+z)^{-1}(x^{-1}+y^{-1}+z^{-1})(xy+yz+xz)^{-1}((xy)^{-1}+(yz)^{-1}+(xz)^{-1}) =$$

$$\frac{1}{x+y+x}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\frac{1}{xy+yz+zx}\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{xz}\right) =$$

$$\frac{1}{x+y+z}\cdot\frac{xy+yz+xz}{xyz}\cdot\frac{1}{xy+yz+xz}\cdot\frac{x+y+z}{xyz} = \frac{1}{x^2y^2z^2} = x^{-2}y^{-2}z^{-2}$$

2. A woman, her brother, her son and her daughter (all relations by birth) are chess players. The worst player's twin (who is one of the four players) and the best player are of opposite sex. The worst player and the best player are of the same age. Who is the worst player?

(A) the woman (B) her brother **(C) her son** (D) her daughter
(E) No solution is consistent with the given information

**Solution:** (C) If the son is the worst player, the daughter must be his twin. The best player is the woman's brother. This is consistent with the given information; the woman's brother and her son could be the same age. In all other cases we get that the woman is the same age as either her son or her daughter which is not possible.

3. The number of points common to the graphs of $(x-y+2)(3x+y-4)=0$ and $(x+y-2)(2x-5y+7)=0$ is:

(A) 2 **(B) 4** (C) 6 (D) 8 (E) None of the answers (A) through (D) is correct.

**Solution:** (B) Each of the graphs is a pair of nonparallel lines. Each line of the first pair intersect the second pair in two distinct points, giving a total of four common points.

4. Let $a$, $b$, $c$, and $d$ be nonzero real numbers such that $c$ and $d$ are solutions of $x^2+ax+b=0$ and $a$ and $b$ are solutions of $x^2+cx+d=0$. Determine the value of $a+b+c+d$.

(A) 0 (B) 2 (C) 4 **(D) -2** (E) None of the answers (A) through (D) is correct.

**Solution:** (D) We have $a+b=-c, c+d=-a, ab=d, cd=b$. From the first two equations we get $b=d$, which implies $a=c=1$. Then $b=d=-2$ and $a+b+c+d=-2$.

5. A positive integer $n$ not exceeding 100 is chosen in a such a way that if $n \le 50$, then the probability of choosing $n$ is $p$, and if $n > 50$, then the probability of choosing $n$ is $3p$. Find the probability that a perfect square is chosen.

(A) 0.05 (B) 0.065 **(C) 0.08** (D) 0.09 (E) 0.1

**Solution:** (C) From $50p + 50(3p) = 1$ we get $p = 0.005$. There are seven perfect squares less than 50, and three perfect squares greater between 50 and 100. Hence, the probability that a perfect square is chosen is $7 \cdot 0.005 + 3 \cdot 3 \cdot 0.005 = 0.08$.

6. In the sequence of numbers $1, 3, 2, \ldots$ each term after the first two is equal to the term preceding it minus the term preceding that. The sum of the first two hundred terms is

(A) 2 (B) -1 (C) 0 **(D) 4** (E) 5

**Solution:** (D) The first eight terms of the sequence are: 1,3,2,-1,-3,-2,1,3. The seventh and the eight terms are the same as the first and the second respectively; hence the sequence repeats every six terms. The sum of the first 198 terms is 0. Therefore the sum of the first two hundred terms is the same as the sum of the 199th and 200th terms, i.e. $1+3=4$.

7. An arbitrary circle can intersect the graph of $y = \sin x$ in:

(A) at most 2 points (B) at most 4 points (C) at most 6 points (D) at most 8 points **(E) None of the answers (A) through (D) is correct.**

**Solution:** (E) Consider a circle tangent to the $x$-axis at the origin. If the radius of the circle is very large, the circle stays close to the $x$-axis. So by making the radius of the circle arbitrary large, the number of intersection points may be made arbitrary large.

8. The volume of a certain rectangular solid is 8 $\text{cm}^3$. Its total surface area is 32 $\text{cm}^2$, and its three dimensions are in geometric progression. The sum of the lengths, in cm, of all the edges of the solid is

(A) 28 **(B) 32** (C) 36 (D) 40 (E) None of the answers (A) through (D) is correct.

**Solution:** (B) Label the edges of the solid by $a$, $ar$, and $ar^2$. Since the volume of the solid is 8, we get $a^3r^3 = 8$, i.e. $ar = 2$. The surface area is 32, so we get $2a^2r + 2a^2r^2 + 2a^2r^3 = 32$, i.e. $2ar(a+ar+ar^2) = 32$. Since $ar = 2$ we get $4(a+ar+ar^2) = 32$. However, $4(a+ar+ar^2)$ represents the sum of the lengths of all sides of the solid.

9. If $\tan\alpha$ and $\tan\beta$ are the roots of $x^2 - px + q = 0$, and $\cot\alpha$ and $\cot\beta$ are the roots of $x^2 - rx + s = 0$, then $rs$ is

(A) $pq$ (B) $\frac{1}{pq}$ **(C)** $\frac{p}{q^2}$ (D) $\frac{q}{p^2}$ (E) $\frac{p}{q}$

**Solution:** (C) It follows that $p = \tan\alpha + \tan\beta$, $q = \tan\alpha\tan\beta$, $r = \cot\alpha + \cot\beta$, $s = \cot\alpha\cot\beta$. Since $\cot\alpha + \cot\beta = \frac{1}{\tan\alpha} + \frac{1}{\tan\beta} = \frac{\tan\alpha+\tan\beta}{\tan\alpha\tan\beta}$ and $\cot\alpha\cot\beta = \frac{1}{\tan\alpha\tan\beta}$, we get $r = \frac{p}{q}$ and $s = \frac{1}{q}$. Hence, $rs = \frac{p}{q^2}$.

10. The ratio of the interior angles of two regular polygons with sides of length 1 in is $3:2$. How many such pairs are there?

    (A) 1 (B) 2 **(C) 3** (D) 4 (E) (F) None of the answers (A) through (D) is correct.

**Solution:** (C) Let $m$ and $n$ be the number of the sides of the two polygons. Then $\frac{(m-2)180}{m} = \frac{3(n-2)180}{2n}$. Hence, $m = \frac{4n}{6-n}$. The possible values for $n$ are 3,4, or 5. If $n = 3$, then $m = 4$; if $n = 4$, then $m = 8$; if $n = 5$, then $m = 20$. So there are three pairs of polygons with the given property.

11. In counting $n$ colored balls, some red and some black, it was found that 49 out of the first 50 counted were red. Thereafter, 7 out of every 8 counted were red. If, in all, 90% or more of the balls counted were red, the maximum value of $n$ is:

    (A) 225 (B) 200 (C) 180 **(D)** 210 (E) 175

**Solution:** (D) Since $\dfrac{49 + \frac{7}{8}(n-50)}{n} \geq \dfrac{9}{10}$, we get $n \leq 210$.

12. Let $a$, $b$, and $c$ be real numbers such that $a \neq b \neq c \neq a$. Find the number of solutions of the equation

$$a^2 \cdot \frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2 \cdot \frac{(x-c)(x-a)}{(b-c)(b-a)} + c^2 \cdot \frac{(x-a)(x-b)}{(c-a)(c-b)} = x^2$$

    (A) 0 (B) 1 (C) 2 **(D) infinitely many**
    (E) None of the answers (A) through (D) is correct.

**Solution:** (D) We see that $a$, $b$, and $c$ are distinct solutions of the given quadratic equation. However, if a quadratic equation has more that three distinct roots, every real number is a solution of the equation.

13. Let $a, b, c, x$ be real numbers for which $\log_a x, \log_b x, \log_c x$ are defined. If the numbers $\log_a x, \log_b x, \log_c x$ form an arithmetic progression (in the given order) and $x \neq 1$, then $c^2$ is equal to:

    (A) $(ab)^{\log_a b}$ (B) $(bc)^{\log_a b}$ **(C)** $(ac)^{\log_a b}$ (D) $(ab)^{\log_c b}$
    (E) None of the answers (A) through (D) is correct.

**Solution:** (C) Since $2\log_b x = \log_a x + \log_c x$, we get $2\frac{\log_a x}{\log_a b} = \log_a x + \frac{\log_a x}{\log_a c}$. Since $x \neq 1$, then $\log_a x \neq 0$, so $\frac{2}{\log_a b} = 1 + \frac{1}{\log_a c}$, i.e. $2\log_a c = \log_a b(\log_a c + 1) = \log_a b(\log_a c + \log_a a) = \log_a b(\log_a(ac)$. Hence $\log_a(c^2) = \log_a(ac)^{\log_a b}$. Therefore, $c^2 = (ac)^{\log_a b}$

14. Find the number of pairs $(m, n)$ of integer numbers which satisfy the equation

$$m^3 + 6m^2 + 5m = 27n^3 + 9n^2 + 9n + 1.$$

**(A) 0** (B) 1 (C) 2 (D) 3 (E) infinitely many

**Solution:** (A) We can transform the equation in the following form

$$m(m+1)(m+5) = 3(9n^3 + 3n^2 + 3n) + 1.$$

The left hand side of the last equation is divisible by 3 and the right hand side is not divisible by 3. Therefore, there are no pairs $(m, n)$ of integer numbers that satisfy the given equation.

15. A number $n$ has three digits when expressed in base 7. When $n$ is expressed in base 9 the digits are reversed. Then the middle digit is

**(A) 0** (B) 2 (C) 4 (D) 6 (E) None of the answers (A) through (D) is correct.

**Solution:** (A) Let $a_1, a_2, a_3$ be the digits of the number $n$. Then $a_1 \cdot 7^2 + a_1 \cdot 7 + a_3 = a_3 \cdot 9^2 + a_2 \cdot 9 + a_3$. Hence $3a_1 = 5a_3 + \frac{a_2}{8}$. Since $a_2 < 7$, we get $a_2 = 0$. From $3a_1 = 5a_3$, we get $a_1 = 5$ and $a_3 = 3$, so $n$ in base 7 is 503 and in base 9 is 305.

16. The function $f$ is not defined for $x = 0$. For all nonzero numbers $x$, $f(x) + 2f\left(\frac{1}{x}\right) = 3x$. The equation $f(x) = f(-x)$ is satisfied by

(A) exactly one real number **(B) exactly two real numbers** (C) no real numbers
(D) all nonzero real numbers (E) None of the answers (A) through (D) is correct.

**Solution:** (B) Replacing $x$ by $\frac{1}{x}$ in the given equation we get $f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x}$. By eliminating $f\left(\frac{1}{x}\right)$ from the last equation and the given equation we get $f(x) = \frac{2-x^2}{x}$. The equation $f(x) = f(-x)$ is satisfied if and only if $\frac{2-x^2}{x} = \frac{2-(-x)^2}{-x}$, i.e. $x^2 = 2$. Hence, $x = \pm\sqrt{2}$ are the only two solutions of this equation.

17. Find the sum of all solutions of the equation

$$\log_{(3x+7)}(4x^2 + 12x + 9) + \log_{(2x+3)}(6x^2 + 23x + 21) = 4$$

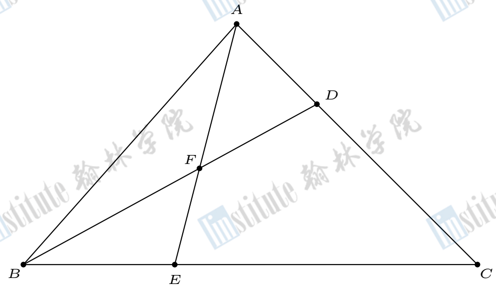
(A) $-4$ (B) $-\frac{25}{4}$ (C) $-\frac{17}{4}$ **(D) $-\frac{1}{4}$** (E) None of the answers (A) through (D) is correct.

**Solution:** (D) The equation is equivalent to $\log_{(3x+7)}(2x+3)^2 + \log_{(2x+3)}(2x+3)(3x+7) = 4$. Since $3x+7 > 0$, $2x+3 > 0$, $3x+7 \neq 1$, and $2x+3 \neq 1$, the domain of the equation is $\left(-\frac{3}{2}, -1\right) \cup (-1, \infty)$. Then we get

$$2\left(\log_{(3x+7)}(2x+3)\right)^2 - 3\log_{(3x+7)}(2x+3) + 1 = 0$$

Then $\log_{(3x+7)}(2x+3) = 1$ or $\log_{(3x+7)}(2x+3) = \frac{1}{2}$ which imply $x_1 = -4$, $x_2 = -2$, $x_3 = -\frac{1}{4}$. Since $-4$ and $-2$ are not in the domain of the equation, the only solution of the equation is $x = -\frac{1}{4}$.

18. In triangle $ABC$, point $D$ divides side $\overline{AC}$ in the ratio $1:2$ ($AD : DC = 1:2$). Let $E$ be the point of intersection of $\overline{BC}$ and $\overline{AF}$, where $F$ is the midpoint of $\overline{BD}$. Find the ratio $BE : EC$.




(A) $1:2$ (B) $3:2$ **(C) $1:3$** (D) $3:1$ (E) None of the answers (A) through (D) is correct.

**Solution:** (C) Draw $\overline{DG}$ parallel to $\overline{AE}$. Then $BE = EG$ because $BF = FD$. Also $2EG = GC$ because $2AD = DC$. Hence $3BE = EG + GC = EC$.

Therefore $E$ divides $BC$ in the ratio of $1:3$.

19. The product $(1+\tan 1^\circ)(1+\tan 2^\circ)\cdots(1+\tan 43^\circ)(1+\tan 44^\circ)$ is equal to

(A) $\tan 1^\circ \tan 2^\circ \cdots \tan 43^\circ \tan 44^\circ$ (B) $(\sqrt{2})^{22}$ (C) $\left(\frac{3}{2}\right)^{44}$ **(D) $2^{22}$**
(E) None of the answers (A) through (D) is correct.

**Solution:** (D) Let $\alpha$ be an acute angle in a right triangle less than $45^\circ$. Then $1+\tan\alpha = \tan 45^\circ + \tan\alpha = \tan(45^\circ+\alpha)(1-\tan 45^\circ \cdot \tan\alpha) = \frac{\sin(45^\circ+\alpha)}{\cos(45^\circ+\alpha)} \cdot \frac{\cos\alpha - \sin\alpha}{\cos\alpha} = \frac{\sin(45^\circ+\alpha)(\cos\alpha-\sin\alpha)}{(\cos 45^\circ \cos\alpha - \sin 45^\circ \sin\alpha)\cos\alpha} = \frac{\sqrt{2}\sin(45^\circ+\alpha)}{\cos\alpha} = \sqrt{2}$. Hence, $(1+\tan 1^\circ)(1+\tan 2^\circ)\cdots(1+\tan 43^\circ)(1+\tan 44^\circ) = (\sqrt{2})^{44} = 2^{22}$.

20. Three men, Adam, Josh, and Kasey, working together, do a job in 6 hours less time than Adam alone, in 1 hour less time than Josh alone, and in one-half the time needed by Kasey when working alone. The number of hours needed by Adam and Josh, working together, to do the job, is:

(A) $\frac{5}{2}$ (B) $\frac{3}{2}$ **(C) $\frac{4}{3}$** (D) $\frac{5}{4}$ (E) $\frac{3}{4}$

**Solution:** (C) Let $a$, $j$, and $k$ be the number of hours needed by Adam, Josh, and Kasey, respectively, to do the job when working alone. Then $\frac{1}{a}+\frac{1}{j}+\frac{1}{k} = \frac{1}{a-6} = \frac{1}{j-1} = \frac{1}{\frac{k}{2}}$. Then $j = a-5$, $k = 2a-12$. We get $\frac{1}{a}+\frac{1}{a-5}+\frac{1}{2a-12} = \frac{1}{a-6}$. The solutions of the last equation are $a = \frac{20}{3}$ and $a = 3$; we reject the second solution because the first hypothesis does not make sense for $a = 3$. Then $j = \frac{5}{3}$ and $k = \frac{4}{3}$. We are looking for $\frac{1}{a}+\frac{1}{j}$ which is $\frac{4}{3}$.

## PART II: 10 INTEGER ANSWER PROBLEMS

1. Let $x_1 = 97$, and let $x_n = \frac{n}{x_{n-1}}$ for $n > 1$. Determine the product $x_1 x_2 \cdots x_8 x_{10}$.

**Solution:** Answer: 3840. For $n > 1$, $x_{n-1}x_n = n$. Hence $x_1x_2 \cdots x_8x_{10} = (x_1x_2)(x_3x_4)(x_5x_6)(x_7x_8)(x_9x_{10}) = 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 = 3840$.

2. When a right triangle is rotated about one leg, the volume of the cone obtained is $800\pi$ cm$^3$. When the triangle is rotated about the other leg, the volume of the cone produced is $1920\pi$ cm$^3$. What is the length, in centimeters, of the hypotenuse of the triangle?

**Solution:** Answer: 26. Let $a$ and $b$ be the lengths of of the legs (in cm) of the right triangle. Then $\frac{\pi}{3}ba^2 = 800\pi$ and $\frac{\pi}{3}ab^2 = 1920\pi$ Hence $a^2b = 2400$ and $ab^2 = 5760$. Dividing the first equation by the second, we get $\frac{a}{b} = \frac{5}{12}$, i.e. $a = \frac{5}{12}b$. Multiplying the last equation by $a^2$, we get $a^3 = \frac{5}{12}a^2b = \frac{5}{12} \cdot 2400 = 1000$. Hence, $a = 10$ and $b = 24$. By Pythagorean Theorem, the length (in cm) of the hypotenuse is 26.

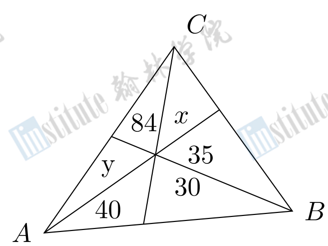
3. A teenage boy wrote his own age after his father's. From this new four digit number he subtracted the absolute value of the difference of their ages to get 4289. Find the sum of their ages.

**Solution:** Answer: 59. Let $f = \overline{f_1f_2}$ and $b = \overline{b_1b_2}$ be the ages of the father and the son respectively. Then $\overline{f_1f_2b_1b_2} - |\overline{f_1f_2} - \overline{b_1b_2}| = 4289$, i.e. $100\overline{f_1f_2} + \overline{b_1b_2} - \overline{f_1f_2} + \overline{b_1b_2} = 4289$. The last equation implies $99\overline{f_1f_2} + 2\overline{b_1b_2} = 4289$, i.e. $99f + 2b = 4289$. Since $13 \le b \le 19$, we see that $2b$ is equal to the remainder in the division of 4289 by 99. Hence, $2b = 32$, i.e. $b = 16$. Then $f = 43$. Therefore, $f + b = 59$.

4. Find the least positive integer $n$ for which $\frac{n-13}{5n+6}$ is a nonzero reducible fraction.

**Solution:** Answer: 84. $\frac{n-13}{5n+6}$ is reducible and nonzero if and only if its reciprocal $\frac{5n+6}{n-13}$ is reducible. Using long division, we get $\frac{5n+6}{n-13} = 5 + \frac{71}{n-13}$ and we conclude that $\frac{71}{n-13}$ must be reducible. Since 71 is prime, $n - 13$ must be divisible by 71. Therefore, the smallest positive $n$ for which 71 divides $n - 13$ is 84.

5. As shown in the figure, $\triangle ABC$ is divided into six smaller triangles by lines drown from the vertices through a common interior point. The areas (in cm$^2$) of the four of these triangles are as indicated. Find the area (in cm$^2$) of $\triangle ABC$.




**Solution:** Answer: 315. If two triangles have the same height, then their areas are proportional to their bases. Hence $\frac{40}{30} = \frac{40+y+84}{30+35+x}$, $\frac{35}{x} = \frac{35+30+40}{x+84+y}$, and $\frac{84}{y} = \frac{84+x+35}{y+40+30}$. The solution of this system is $x = 70$ and $y = 56$. Therefore, the area of the triangle $ABC$ is 315.

6. Find the minimum value of

$$f(x) = \frac{9x^2 \sin^2 x + 4}{x \sin x}$$

for $0 < x < \pi$.

**Solution:** Answer: 12. We can re-write $f(x)$ as $f(x) = 9x\sin x + \frac{4}{x\sin x}$. Applying the arithmetic-geometric mean inequality gives $f(x) \geq 12$ and equality holds if and only of $9x\sin x = \frac{4}{x\sin x}$, i.e. $x^2\sin^2 x = \frac{4}{9}$. Since $0^2\sin^2 0 = 0$ and $\left(\frac{\pi}{2}\right)^2 \sin^2\frac{\pi}{2} > 1$, it follows that there is $x_0 \in \left(0, \frac{\pi}{2}\right)$ such that $x_0^2 \sin^2 x_0 = \frac{4}{9}$.

7. Let $a_n = 6^n + 8^n$. Determine the remainder on dividing $a_{83}$ by 49.

**Solution:** Answer: 35. For odd $n$ we have

$$a_n = (7-1)^n + (7+1)^n = 2\left(7^n + \binom{n}{2}7^{n-2} + \cdots + \binom{n}{n-3}7^3 + \binom{n}{n-1}7\right)$$

Hence, for $n$ odd, $a_n = 2\cdot 49\left(7^{n-2} + \binom{n}{2}7^{n-4} + \cdots + \binom{n}{n-3}7^3\right) + 14n$. Therefore $a_{83} = 49q + 1162 = 49(q+23) + 35$. Therefore, the remainder is 35.

8. What is the smallest positive odd integer $n$ such that the product

$$2^{\frac{1}{7}} \cdot 2^{\frac{3}{7}} \cdots 2^{\frac{2n+1}{7}}$$

is greater than 1000?

**Solution:** Answer: 9. Since $2^{\frac{1}{7}} \cdot 2^{\frac{3}{7}} \cdots 2^{\frac{2n+1}{7}} = 2^{\frac{1+3+\cdots(2n+1)}{7}} = 2^{\frac{(n+1)^2}{7}}$ and $2^{10} = 1024$ we have:

$$2^{\frac{(7+1)^2}{7}} = 2^{9+\frac{1}{7}} < 2^9 \cdot 2^{\frac{1}{2}} < 1000 < 2^{10} < 2^{\frac{(9+1)^2}{7}}$$

9. Find the sum of the squares of all real solutions of

$$\sqrt[4]{13+x} + \sqrt[4]{4-x} = 3$$

**Solution:** Answer: 153. Let $u = \sqrt[4]{13+x}$ and $v = \sqrt[4]{4-x}$. Then $u + v = 3$ and $u^4 + v^4 = 17$. Hence,

$$17 + 2u^2v^2 = (u^2+v^2)^2 = ((u+v)^2 - 2uv)^2 = (9-2uv)^2.$$

Then we get $u^2v^2 - 18uv + 32 = 0$, i.e $(uv-2)(uv-16) = 0$. If $uv = 2$, then $(13+x)(4-x) = 16$. Hence, $x = 3$ or $x = -12$. If $uv = 16$, then $(13+x)(4-x) = 16^4$, i.e. $x^2 + 9x + 16^4 = 0$ which does not have real solution. Therefore, the sum of the squares of all real solution of the equation are $3^2 + (-12)^2 = 153$.

10. Find the smallest positive integer $n$ such that

$$(x^2+y^2+z^2)^2 \leq n(x^4+y^4+z^4)$$

for all real numbers $x$, $y$, and $z$.

**Solution:** Answer: 3. Let $a = x^2$, $b = y^2$, $c = z^2$. Then $(a-b)^2 + (b-c)^2 + (a-c)^2 \geq 0$. This implies $\frac{ab+bc+ac}{a^2+b^2+ac} \leq 1$. Then $\frac{a^2+b^2+c^2}{a^2+b^2+c^2} + \frac{2(ab+bc+ac)}{a^2+b^2+c^2} \leq 1+2 = 3$. Hence $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$. The inequality holds if $n = 3$. To eliminate $n = 1$ and $n = 2$, we consider the case $a = b = c > 0$. Therefore, the smallest value of $n$ for which the inequality holds is 3.

The following problem, will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test.

# TIE BREAKER PROBLEM

Let $a$, $b$, $c$, $a+b-c$, $a+c-b$, $b+c-a$, $a+b+c$ be 7 distinct prime numbers such that the sum of two of $a$, $b$, $c$ is 800. Let $d$ be the difference between the largest and the smallest numbers among the 7 primes. Find the largest possible value of $d$.

**Solution:** Answer: 1594. We first note that if one of the numbers $a$, $b$, $c$ is 2, say $a = 2$, then $b$ and $c$ are odd, so $a+b+c$ is even and greater than 2, which contradicts the hypothesis that it is a prime. Thus, $a, b, c \geq 3$ and all seven primes are odd. Without loss of generality, we may assume that $a+b = 800$ and $a < b$. Since $a+b-c \geq 3$, we get that $c \leq 797$. The greatest of the seven primes is $a+b+c$. Therefore,

$$d \leq (a+b+c) - 3 \leq 800 + 797 - 3 = 1594$$

The maximum value of $d$ is 1594: for example, if $a = 13$, $b = 787$, $c = 797$, $a+b+c = 1597$, $a+b-c = 23$, $a+c-b = 3$ and $b+c-a = 1571$ (all seven numbers are prime!), we get that $d = 1594$.