The Thirty-fifth Annual
State High School
Mathematics Contest

Thursday, April 24, 2014

Held on the Campus
of the North Carolina School
of Science and Mathematics
Durham, NC

Sponsored by
The North Carolina Council
of Teachers of Mathematics

NC STATE MATHEMATICS CONTEST
APRIL 2014

PART I: 20 MULTIPLE CHOICE PROBLEMS

1. For all positive real numbers x, y, z, the product

$$(x+y+z)^{-1}(x^{-1}+y^{-1}+z^{-1})(xy+yz+xz)^{-1}((xy)^{-1}+(yz)^{-1}+(xz)^{-1})$$

is equal to

(A) $\frac{1}{xyz}$ (B) $x^{-2}y^{-2}z^{-2}$ (C) $(x+y+z)^{-1}$ (D) $x^{-2}+y^{-2}+z^{-2}$
(E) None of the answers (A) through (D) is correct.

2. A woman, her brother, her son and her daughter (all relations by birth) are chess players. The worst player's twin (who is one of the four players) and the best player are of opposite sex. The worst player and the best player are of the same age. Who is the worst player?

(A) the woman (B) her brother (C) her son (D) her daughter
(E) No solution is consistent with the given information

3. The number of points common to the graphs of $(x-y+2)(3x+y-4)=0$ and $(x+y-2)(2x-5y+7)=0$ is:

(A) 2 (B) 4 (C) 6 (D) 8 (E) None of the answers (A) through (D) is correct.

4. Let a, b, c, and d be nonzero real numbers such that c and d are solutions of $x^2+ax+b=0$ and a and b are solutions of $x^2+cx+d=0$. Determine the value of $a+b+c+d$.

(A) 0 (B) 2 (C) 4 (D) -2 (E) None of the answers (A) through (D) is correct.

5. A positive integer n not exceeding 100 is chosen in a such a way that if $n \leq 50$, then the probability of choosing n is p, and if $n > 50$, then the probability of choosing n is $3p$. Find the probability that a perfect square is chosen.

(A) 0.05 (B) 0.065 (C) 0.08 (D) 0.09 (E) 0.1

6. In the sequence of numbers $1, 3, 2, \ldots$ each term after the first two is equal to the term preceding it minus the term preceding that. The sum of the first two hundred terms is

(A) 2 (B) -1 (C) 0 (D) 4 (E) 5

7. An arbitrary circle can intersect the graph of $y = \sin x$ in:

(A) at most 2 points (B) at most 4 points (C) at most 6 points (D) at most 8 points
(E) None of the answers (A) through (D) is correct.

8. The volume of a certain rectangular solid is 8 cm^3. Its total surface area is 32 cm^2, and its three dimensions are in geometric progression. The sum of the lengths, in cm, of all the edges of the solid is

(A) 28 (B) 32 (C) 36 (D) 40 (E) None of the answers (A) through (D) is correct.

9. If $\tan\alpha$ and $\tan\beta$ are the roots of $x^2 - px + q = 0$, and $\cot\alpha$ and $\cot\beta$ are the roots of $x^2 - rx + s = 0$, then rs is

(A) pq (B) $\frac{1}{pq}$ (C) $\frac{p}{q^2}$ (D) $\frac{q}{p^2}$ (E) $\frac{p}{q}$

10. The ratio of the interior angles of two regular polygons with sides of length 1 in is 3 : 2. How many such pairs are there?

(A) 1 (B) 2 (C) 3 (D) 4 (E) (F) None of the answers (A) through (D) is correct.

11. In counting n colored balls, some red and some black, it was found that 49 out of the first 50 counted were red. Thereafter, 7 out of every 8 counted were red. If, in all, 90% or more of the balls counted were red, the maximum value of n is:

(A) 225 (B) 200 (C) 180 (D) 210 (E) 175

12. Let a, b, and c be real numbers such that $a \neq b \neq c \neq a$. Find the number of solutions of the equation

$$a^2 \cdot \frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2 \cdot \frac{(x-c)(x-a)}{(b-c)(b-a)} + c^2 \cdot \frac{(x-a)(x-b)}{(c-a)(c-b)} = x^2$$

(A) 0 (B) 1 (C) 2 (D) infinitely many
(E) None of the answers (A) through (D) is correct.

13. Let a, b, c, x be real numbers for which $\log_a x, \log_b x, \log_c x$ are defined. If the numbers $\log_a x, \log_b x, \log_c x$ form an arithmetic progression (in the given order) and $x \neq 1$, then c^2 is equal to:

(A) $(ab)^{\log_a b}$ (B) $(bc)^{\log_a b}$ (C) $(ac)^{\log_a b}$ (D) $(ab)^{\log_c b}$
(E) None of the answers (A) through (D) is correct.

14. Find the number of pairs (m, n) of integer numbers which satisfy the equation

$$m^3 + 6m^2 + 5m = 27n^3 + 9n^2 + 9n + 1.$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) infinitely many

15. A number n has three digits when expressed in base 7. When n is expressed in base 9 the digits are reversed. Then the middle digit is

(A) 0 (B) 2 (C) 4 (D) 6 (E) None of the answers (A) through (D) is correct.

16. The function f is not defined for $x = 0$. For all nonzero numbers x, $f(x) + 2f\left(\frac{1}{x}\right) = 3x$. The equation $f(x) = f(-x)$ is satisfied by

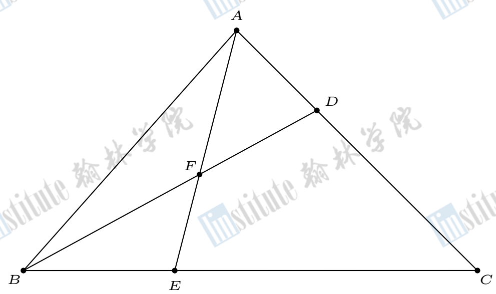
(A) exactly one real number (B) exactly two real numbers (C) no real numbers
(D) all nonzero real numbers (E) None of the answers (A) through (D) is correct.

17. Find the sum of all solutions of the equation

$$\log_{(3x+7)}(4x^2+12x+9)+\log_{(2x+3)}(6x^2+23x+21)=4$$

(A) -4 (B) $-\frac{25}{4}$ (C) $-\frac{17}{4}$ (D) $-\frac{1}{4}$ (E) None of the answers (A) through (D) is correct.

18. In triangle ABC, point D divides side $\overline{AC}$ in the ratio $1:2$ ($AD:DC=1:2$). Let E be the point of intersection of $\overline{BC}$ and $\overline{AF}$, where F is the midpoint of $\overline{BD}$. Find the ratio $BE:EC$.



(A) $1:2$ (B) $3:2$ (C) $1:3$ (D) $3:1$ (E) None of the answers (A) through (D) is correct.

19. The product $(1+\tan 1°)(1+\tan 2°)\cdots(1+\tan 43°)(1+\tan 44°)$ is equal to

(A) $\tan 1°\tan 2°\cdots\tan 43°\tan 44°$ (B) $(\sqrt{2})^{22}$ (C) $\left(\frac{3}{2}\right)^{44}$ (D) 2^{22}
(E) None of the answers (A) through (D) is correct.

20. Three men, Adam, Josh, and Kasey, working together, do a job in 6 hours less time than Adam alone, in 1 hour less time than Josh alone, and in one-half the time needed by Kasey when working alone. The number of hours needed by Adam and Josh, working together, to do the job, is:

(A) $\frac{5}{2}$ (B) $\frac{3}{2}$ (C) $\frac{4}{3}$ (D) $\frac{5}{4}$ (E) $\frac{3}{4}$

PART II: 10 INTEGER ANSWER PROBLEMS

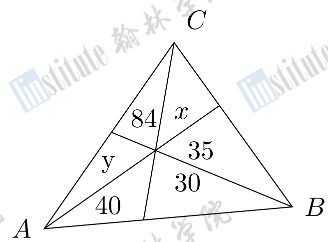
1. Let $x_1=97$, and let $x_n=\frac{n}{x_{n-1}}$ for $n>1$. Determine the product $x_1x_2\cdots x_8x_{10}$.

2. When a right triangle is rotated about one leg, the volume of the cone obtained is 800π cm^3. When the triangle is rotated about the other leg, the volume of the cone produced is 1920π cm^3. What is the length, in centimeters, of the hypotenuse of the triangle?

3. A teenage boy wrote his own age after his father's. From this new four digit number he subtracted the absolute value of the difference of their ages to get 4289. Find the sum of their ages.

4. Find the least positive integer n for which $\frac{n-13}{5n+6}$ is a nonzero reducible fraction.

5. As shown in the figure, $\triangle ABC$ is divided into six smaller triangles by lines drown from the vertices through a common interior point. The areas (in cm^2) of the four of these triangles are as indicated. Find the area (in cm^2) of $\triangle ABC$.



6. Find the minimum value of

$$f(x) = \frac{9x^2 \sin^2 x + 4}{x \sin x}$$

for $0 < x < \pi$.

7. Let $a_n = 6^n + 8^n$. Determine the remainder on dividing a_{83} by 49.

8. What is the smallest positive odd integer n such that the product

$$2^{\frac{1}{7}} \cdot 2^{\frac{3}{7}} \cdots 2^{\frac{2n+1}{7}}$$

is greater than 1000?

9. Find the sum of the squares of all real solutions of

$$\sqrt[4]{13+x} + \sqrt[4]{4-x} = 3$$

10. Find the smallest positive integer n such that

$$(x^2 + y^2 + z^2)^2 \leq n(x^4 + y^4 + z^4)$$

for all real numbers x, y, and z.

The following problem, will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test.

TIE BREAKER PROBLEM

Let a, b, c, $a+b-c$, $a+c-b$, $b+c-a$, $a+b+c$ be 7 distinct prime numbers such that the sum of two of a, b, c is 800. Let d be the difference between the largest and the smallest numbers among the 7 primes. Find the largest possible value of d.