## The Thirty-fifth Annual State High School Mathematics Contest

multitute mark 's

multile # # 3 PS

multilite # # 13 PR

multille \$ 75 'S

multile # # 'S R

mouthing the the 'S PR

Inditute # # 'S R

1. # 14- 13 9%

multitute \$ 75 '3

而此此此恭求长鉴死

而此此此恭求长鉴死

multilite # # '& P&

而此此此新祥生

mutilite # # '& R

Inditute # # 'S R

面动机机精带样谱常

Institute ## # '& R

to the W- B PR

multitute m # 3

Institute # # '& PR

multille # # 'S PE

面机机机新林等席

Institute # # '& PR

面机机机新林塔梯

Institute # # '& R

Institute ## # '& PS

Institute # # '& P&

to the the the

Ro

R

N.

N.

R.

Ro

N.

R.

Ro

multitute m # 3

面动机机都带样等席

multilite # # 13 PR

Institute # # '& R

to the We B

multitute m # 3

multine # # 3 PR

Institute # # 'S PR

面动机机都林塔梯

\*\*\*\*\*\*\*\*\*\*

Thursday, April 24, 2014

Held on the Campus of the North Carolina School of Science and Mathematics Durham, NC

Sponsored by The North Carolina Council of Teachers of Mathematics

Invitute # # '& R

to the the the

## NC STATE MATHEMATICS CONTEST APRIL 2014

tinstitute ##

mistilute # # 'S

Withthe State of the State of t

物於

## Astitute the the PART I: 20 MULTIPLE CHOICE PROBLEMS

1. For all positive real numbers x, y, z, the product

 $(x+y+z)^{-1}(x^{-1}+y^{-1}+z^{-1})(xy+yz+xz)^{-1}((xy)^{-1}+(yz)^{-1}+(xz)^{-1})$ is equal to (A)  $\frac{1}{xyz}$  (B)  $x^{-2}y^{-2}z^{-2}$  (C)  $(x+y+z)^{-1}$  (D)  $x^{-2}+y^{-2}+z^{-2}$ (E) None of the answers (A) through (D) is correct.

minitial the start of

Y.

N.

Y.

N.

Y.

Y.

Y.

\*\*\*\*\*\*

2. A woman, her brother, her son and her daughter (all relations by birth) are chess players. The worst player's twin (who is one of the four players) and the best player are of opposite sex. The worst player and the best player are of the same age. Who is the worst player?

- (A) the woman (B) her brother (C) her son (D) her daughter
- (E) No solution is consistent with the given information

3. The number of points common to the graphs of (x-y+2)(3x+y-4) = 0 and (x+y)(3x+y-4) = 0itute % aturit. is:

(E) None of the answers (A) through (D) is correct.  $(\mathbf{A})$ (B) (C) 6 (D) 8

4. Let a, b, c, and d be nonzero real numbers such that c and d are solutions of  $x^2 + ax + b = 0$  and a and b are solutions of  $x^2 + cx + d = 0$ . Determine the value of a + b + c + d.

(A) 0 (C) 4 (D) -2 (E) None of the answers (A) through (D) is correct. (B) 2

- 5. A positive integer n not exceeding 100 is chosen in a such a way that if  $n \leq 50$ , then the probability of choosing n is p, and if n > 50, then the probability of choosing n is 3p. Find the probability that a perfect square is chosen.
  - (A) 0.05 (C) 0.08 (B) 0.065 (D) 0.09 (E) 0.1
- 6. In the sequence of numbers  $1, 3, 2, \ldots$  each term after the first two is equal to the term preceding it minus the term preceding that. The sum of the first two hundred terms is

(B) -1 (C) 0(D) 4 (E) 5 (A)

 $\sqrt{7}$ . An arbitrary circle can intersect the graph of  $y = \sin x$  in:

- (A) at most 2 points (B) at most 4 points (C) at most 6 points (D) at most 8 points (E) None of the answers (A) through (D) is correct.
- 8. The volume of a certain rectangular solid is  $8 \text{ cm}^3$ . Its total surface area is  $32 \text{ cm}^2$ , and its three dimensions are in geometric progression. The sum of the lengths, in cm, of all the edges of the solid is (C) 36 (D) 40 (E) None of the answers (A) through (D) is correct. (B) 32

1

9. If  $\tan \alpha$  and  $\tan \beta$  are the roots of  $x^2 - px + q = 0$ , and  $\cot \alpha$  and  $\cot \beta$  are the roots of  $x^2 - rx + s = 0$ , then rs is

multille m 25 'S

institute ##

2

itute ##

面站曲線

itute ##

Withte # # 13 PR

\*\*\*\*\*\*\*

Institute \$7 77 'S

(D)  $\frac{q}{r^2}$ (C)  $\frac{p}{a^2}$ (E)  $\frac{p}{a}$ (A) pq(B)

Mustitute 300 X

Institute ##

minitial the start of

Y.

N.

Y.

Ro

Y.

Y.

- 10. The ratio of the interior angles of two regular polygons with sides of length 1 in is 3 : 2. How many such pairs are there?
  - (A) 1 (B) 2(D) 4 (E)(F) None of the answers (A) through (D) is correct. (C) 3
- 11. In counting n colored balls, some red and some black, it was found that 49 out of the first 50 counted were red. Thereafter, 7 out of every 8 counted were red. If, in all, 90% or more of the balls counted were red, the maximum value of n is:
  - (A) 225 (B) 200 (C) 180 (D) 210 (E) 175

te We 12. Let a, b, and c be real numbers such that  $a \neq b \neq c \neq a$ . Find the number of solutions of the equation

$$a^{2} \cdot \frac{(x-b)(x-c)}{(a-b)(a-c)} + b^{2} \cdot \frac{(x-c)(x-a)}{(b-c)(b-a)} + c^{2} \cdot \frac{(x-a)(x-b)}{(c-a)(c-b)} = x^{2}$$
  
C) 2 (D) infinitely many  
swers (A) through (D) is correct.

(A) 0(C) 2 (D) infinitely many (B) 1 (E) None of the answers (A) through (D) is correct.

- 13. Let a, b, c, x be real numbers for which  $\log_a x$ ,  $\log_b x$ ,  $\log_c x$  are defined. If the numbers  $\log_a x$ ,  $\log_b x$ ,  $\log_c x$ form an arithmetic progression (in the given order) and  $x \neq 1$ , then  $c^2$  is equal to: anstitute the tot is the
  - (A)  $(ab)^{\log_a b}$ (B)  $(bc)^{\log_a b}$ (C)  $(ac)^{\log_a b}$  (D)  $(ab)^{\log_c b}$ (E) None of the answers (A) through (D) is correct.

14. Find the number of pairs (m, n) of integer numbers which satisfy the equation

 $m^3 + 6m^2 + 5m = 27n^3 + 9n^2 + 9n + 1.$ 

(C) 2(B) 1 (D) 3 (E) infinitely many

15. A number n has three digits when expressed in base 7. When n is expressed in base 9 the digits are reversed. Then the middle digit is

(B) 2(D) 6 (E) None of the answers (A) through (D) is correct.  $(A) 0_{0}$ 

Y.

Y.

to the the B

るな来

- 16. The function f is not defined for x = 0. For all nonzero numbers x,  $f(x) + 2f\left(\frac{1}{x}\right) = 3x$ . The equation f(x) = f(-x) is satisfied by
  - (A) exactly one real number (B) exactly two real numbers (C) no real numbers

to the the B Ph

(D) all nonzero real numbers (E) None of the answers (A) through (D) is correct. withthe ## # 12 1% withthe \$6 \$ 12 stitute # # 'S PK Within the the 'S

stitute # \*\*

\*\*\*\*\*

mutilite # # " Institute \$ 75 'S tinstitute ## # multilite # # \* Institute \$7 \$7 'S multilite # # \* 17. Find the sum of all solutions of the equation Institute # \*\*\* Y.  $\log_{(3x+7)}(4x^2 + 12x + 9) + \log_{(2x+3)}(6x^2 + 23x + 21) = 4$ (A) -4 (B)  $-\frac{25}{4}$  (C)  $-\frac{17}{4}$  (D)  $-\frac{1}{4}$  (E) None of the answers (A) through (D) is correct. 18. In triangle ABC, point D divides side  $\overline{AC}$  in the ratio 1 : 2 (AD : DC = 1 : 2). Let E be the point of tute to the intersection of BC and AF, where F is the midpoint of BD. Find the ratio BE : EC. 面对加根教祥等张 multille # # '3 PS myinne # # 13 PR withit the start of the start o Y. Institute ## (B) 3:2 (C) 1:3 (D) 3:1 (E) None of the answers (A) through (D) is correct. N. (A) 1:2withit the the institute \$ 19. The product  $(1 + \tan 1^\circ)(1 + \tan 2^\circ) \cdots (1 + \tan 43^\circ)(1 + \tan 44^\circ)$  is equal to (A)  $\tan 1^{\circ} \tan 2^{\circ} \cdots \tan 43^{\circ} \tan 44^{\circ}$  (B)  $(\sqrt{2})^{22}$  (C)  $(\frac{3}{2})^{44}$ (D)  $2^{22}$ (E) None of the answers (A) through (D) is correct. Y. 20. Three men, Adam, Josh, and Kasey, working together, do a job in 6 hours less time than Adam alone, in 1 hour less time than Josh alone, and in one-half the time needed by Kasey when working alone. The number of hours needed by Adam and Josh, working together, to do the job, is: (B)  $\frac{3}{2}$  (C)  $\frac{4}{3}$  (D)  $\frac{5}{4}$  (E)  $\frac{3}{4}$ (A)  $\frac{5}{2}$ matilite # # '\$ 1% 14 1/2 VIN 6 80 PART II: 10 INTEGER ANSWER PROBLEMS 1. Let  $x_1 = 97$ , and let  $x_n = \frac{n}{x_{n-1}}$  for n > 1. Determine the product  $x_1 x_2 \cdots x_8 x_{10}$ . 2. When a right triangle is rotated about one leg, the volume of the cone obtained is  $800\pi$  cm<sup>3</sup>. When  $\sim$ the triangle is rotated about the other leg, the volume of the cone produced is  $1920\pi$  cm<sup>3</sup>. What is the length, in centimeters, of the hypotenuse of the triangle? 3. A teenage boy wrote his own age after his father's. From this new four digit number he subtracted the absolute value of the difference of their ages to get 4289. Find the sum of their ages. 4. Find the least positive integer n for which  $\frac{n-13}{5n+6}$  is a nonzero reducible fraction. . ch matinte # # 'S R Withit the the the file Astitute State 13 PS Withte # # 18 PK Autitute ## # 'S Y.

3

to the We B

\*\* 标林 洛

to the We B

to the the the Ph

to the the the Ph

Y.

to the be the Bolton

5. As shown in the figure,  $\triangle ABC$  is divided into six smaller triangles by lines drown from the vertices through a common interior point. The areas  $(in \text{ cm}^2)$  of the four of these triangles are as indicated. Find the area (in  $\text{cm}^2$ ) of  $\triangle ABC$ .

multilite # # "

6. Find the minimum value of

multille m # "

$$f(x) = \frac{9x^2 \sin^2 x + 4}{x \sin x}$$

Institute m # \*

multille m # "

mutule # # C

40

面前加出

mating # # 3 %

Witht the start to per

10 the the 1/2 1/2

84 x No.

35

30

multine m \*\* \*

maximue # # 3 PR

而此此此称样谱像

myinne # # 13 PR

而如此他称林塔梯

matine # # 3 PS

Artitute ## # 18

to the the 'B the

for  $0 < x < \pi$ .

multine m # "

Ro

Y.

Y.

Ro

Y.

N.

Y.

Ro

面的机能称林塔张

to the the B

maximue # # 3 PS 7. Let  $a_n = 6^n + 8^n$ . Determine the remainder on dividing  $a_{83}$  by 49. 8. What is the smallest positive odd integer n such that the product

 $2^{\frac{1}{7}} \cdot 2^{\frac{3}{7}} \cdots 2^{\frac{2n+1}{7}}$ 

is greater than 1000?

9. Find the sum of the squares of all real solutions of tinstitute ## #

solutions of 
$$\sqrt[4]{13+x} + \sqrt[4]{4-x} = 3$$

10. Find the smallest positive integer n such that

$$(x^{2} + y^{2} + z^{2})^{2} \le n(x^{4} + y^{4} + z^{4})$$

for all real numbers x, y, and z.

mythte # # '& K

to the the the

而如此他新祥後 maxinte # # \* \* The following problem, will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test. mistine ## # B

## Autitule ## # 18 TIE BREAKER PROBLEM

Let a, b, c, a + b - c, a + c - b, b + c - a, a + b + c be 7 distinct prime numbers such that the sum of two of a, b, c is 800. Let d be the difference between the largest and the smallest numbers among the 7 primes. matitute # # '3 PS matine # # B Withte the the the Arithte the the the Find the largest possible value of d. withite # 3 withit the star the

4

Withte # # 13 PR

to the We B We

Withte # # 3 PR

to the the the