The Thirty-fourth Annual State High School Mathematics Contest

Thursday, April 11, 2013

Held on the Campus
of the North Carolina School
of Science and Mathematics
Durham, NC

Sponsored by
The North Carolina Council
of Teachers of Mathematics

NC STATE MATHEMATICS CONTEST
APRIL 2013

PART I: 20 MULTIPLE CHOICE PROBLEMS

1. Find the sum $1+2-3-4+5+6-7-8+\cdots+2009+2010-2011-2012+2013$.
   (A) 0 **(B)** 1 (C) 5 (D) -3 (E) None of the answers (A) through (D) is correct.

> **Solution:** (B) $1+2-3-4+5+6-7-8+\cdots+2009+2010-2011-2012+2013 =$
> $(1+2-3-4)+(5+6-7-8)+\cdots+(2009+2010-2011-2012)+2013 = 503\cdot(-4)+2013 = 1$.

2. In $\triangle ABC$, $m(\angle ABC) = 140°$ and the points D and E lie on sides $\overline{AB}$ and $\overline{AC}$, respectively. If lengths AD, DE, EB, and BC are all equal, then the measure of $\angle BAC$ is
   (A) $5°$ (B) $6°$ (C) $7.5°$ (D) $8°$ **(E)** $10°$

> **Solution:** (E) Let $x = m(\angle DAE) = m(\angle DEA)$, $y = m(\angle EDB) = m(\angle EBD)$, and $z = m(\angle BEC) = m(\angle BCE)$. Then $y = 2x$, $z = x+y = 3x$. From $\triangle ABC$ we get $x + m(\angle ABC) + z = 180$. Hence, $x = 10$.

3. The numbers $a-b+2013$, $b-c+2013$, and $c-a+2013$ are three consecutive integer numbers. Find $a-3b+2c$.
   (A) -1 (B) 0 (C) 1 (D) -2 (E) 2

> **Solution:** (A) Let $a-b+2013 = n-1,\ b-c+2013 = n,\ c-a+2013 = n+1$. Then $3\cdot 2013 = 3n$, i.e. $n = 2013$. Hence $a-b=-1$, $b-c=0$, and $c-a=1$. Thus $a=b-1$ and $b=c$ which implies $a-3b+2c = b-1-3b+2b = -1$.

4. A circular disk is divided by $2n$ equally spaced radii ($n>0$) and one secant line. The maximum number of non-overlapping areas into which the disk can be divided is
   (A) $2n+1$ (B) $2n+2$ (C) $3n-1$ (D) $3n$ **(E)** $3n+1$

> **Solution:** (E) A secant can cut across at most $(n+1)$ of the $(2n)$ sectors, dividing each into two parts. The total number of distinct areas is $2n+(n+1) = 3n+1$.

5. If $60^a = 3$ and $60^b = 5$, then $12^{\frac{1-a-b}{2(1-b)}}$ is
   (A) $\sqrt{3}$ **(B)** 2 (C) $\sqrt{5}$ (D) 3 (E) $\sqrt{12}$

> **Solution:** (B) $12 = \frac{60}{5} = \frac{60}{60^b} = 60^{1-b}$. So $12^{\frac{1-a-b}{2(1-b)}} = 60^{\frac{1-a-b}{2}} = \sqrt{\frac{60}{60^a\cdot 60^b}} = \sqrt{\frac{60}{3\cdot 5}} = 2$.

6. Let x and y be positive integer numbers such that $\frac{1}{x} - \frac{1}{y} + \frac{1}{xy} = \frac{2}{5}$. Find $2x - 3y$.

(A) 3 (B) 23 **(C) -11** (D) -2 (E) None of the answers (A) through (D) is correct.

> **Solution:** (C) $\frac{1}{x} - \frac{1}{y} + \frac{1}{xy} = \frac{2}{5}$ is equivalent to $(2x-5)(2y+5) = -15$. Hence, $x = 2$, $y = 5$ and $2x - 3y = -11$.

7. If the sum of all angles except one in a convex polygon is $2190°$, then the number of sides of the polygon is

(A) 12 (B) 14 **(C) 15** (D) 16 (E) 17

> **Solution:** (C) Let n denote the number of sides of the given convex polygon and let x be the number of degrees in the excepted angle. Then $180(n-2) = 2190 + x$. From here we have $n-2 = 12 + \frac{30}{180} + \frac{x}{180}$. Hence, $n = 15$.

8. Let z and w be complex numbers such that $|z| = |w| = |z - w|$. Find $\left(\frac{z}{w}\right)^{2013}$.

(A) -1 (B) 1 (C) $\frac{1}{2} + \frac{\sqrt{3}}{2} \cdot i$ (D) $\frac{1}{2} - \frac{\sqrt{3}}{2} \cdot i$ (E) i

> **Solution:** (A) Let $\frac{z}{w} = u = x + i \cdot y$. Then $|u| = |u - 1| = 1$, which implies $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 1$. Thus, $x = \frac{1}{2}$ and $y = \pm\frac{\sqrt{3}}{2}$, i.e. $u = \frac{1}{2} \pm \frac{\sqrt{3}}{2} \cdot i$. Therefore, $\left(\frac{z}{w}\right)^{2013} = \left(\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\right)^3\right)^{671} = (-1)^{671} = -1$.

9. Let x_1 and x_2 be real roots of the equation $x^2 - x + q = 0$, where q is a real number. For which values of q, the expression $x_1^4 + x_2^4$ has minimum value?

(A) $\frac{1}{4}$ (B) 1 (C) 0 (D) -1 (E) None of the answers (A) through (D) is correct.

> **Solution:** (A) The equation $x^2 - x + q = 0$ has real roots if $1 - 4q \geq 0$, i.e. $q \leq \frac{1}{4}$. Since $x_1 + x_2 = 1$ and $x_1 x_2 = q$, we have $x_1^4 + x_2^4 = \left((x_1 + x_2)^2 - 2x_1x_2\right)^2 - 2(x_1x_2)^2 = (1-2q)^2 - 2q^2 = 2(q-1)^2 - 1$. Hence, $x_1^4 + x_2^4$ is decreasing on $(-\infty, \frac{1}{4}]$, so $x_1^4 + x_2^4$ has minimum for $q = \frac{1}{4}$.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = ax^5 + bx^3 + cx - 7$ where a, b, and c are real numbers. If $f(-5) = 5$, find $f(5)$.

(A) -5 (B) 10 (C) 5 (D) 12 **(E) None of the answers (A) through (D) is correct.**

> **Solution:** (E) Since $f(5) = a \cdot 5^5 + b \cdot 5^3 + c \cdot 5 - 7$ and $f(-5) = a \cdot (-5)^5 + b \cdot (-5)^3 + c \cdot (-5) - 7$, we get $f(5) + f(-5) = -14$. Hence, $f(5) = -19$.

11. A drawer contains 10 socks, 6 red, 4 blue. If 4 socks are pulled from the drawer at random, what is the probability that there are more red socks than blue socks?

(A) $\frac{5}{21}$ **(B) $\frac{19}{42}$** (C) $\frac{1}{2}$ (D) $\frac{3}{5}$ (E) $\frac{31}{42}$

Solution:

(B) There are $\binom{10}{4} = 210$ possible sets of 4 socks. If there are more red socks, there are either 4 or 3 red. There are $\binom{6}{4} + \binom{6}{3} \cdot \binom{4}{1} = 15 + 80 = 95$. $\frac{95}{210} = \frac{19}{42}$.

12. Let

$$U(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$$

For how many distinct positive integers, n, is $U(U(U(U(U(n))))) = 1$?

(A) 2 **(B) 3** (C) 4 (D) 5 (E) 8

Solution:

(B) $U(U(U(U(U(n))))) = 1$, $U(U(U(U(n)))) = 2$, $U(U(U(n))) = 4$, $U(U(n)) = 1$ or 8, $U(n) = 2$, or 16, $n = 4, 5$, or 32. So there are 3 such n's.

13. Two candles of the same length are made of different materials so that one burns out completely at a uniform rate in 3 hours and the other, in 4 hours. At what time P.M. should the candles be lightened so that, at 4:00 P.M., one stub is twice the length of the other?

(A) 1:24 (B) 1:28 **(C) 1:36** (D) 1:40 (E) 1:48

Solution: (C) Let t represent the number of hours before 4:00 P.M. so that at 4:00 P.M., one stub is twice the length of the other. Let l be the original length of the candles. Hence, $l - \frac{t}{4}l = 2\left(l - \frac{t}{3}l\right)$. Thus, $t = 2\frac{2}{5}$. Therefore, the candles should be lighted $4 - 2\frac{2}{5} = 1\frac{3}{5}$ hours after noon, i.e. at 1:36 P.M.

14. Evaluate the sum

$$\sin^2(1°)\cos^2(1°) + \sin^2(2°)\cos^2(2°) + \sin^2(3°)\cos^2(3°) + \ldots + \sin^2(44°)\cos^2(44°) + \sin^2(45°)\cos^2(45°)$$

(A) $\frac{23}{8}$ (B) $\frac{45}{8}$ **(C) $\frac{23}{4}$** (D) $\frac{45}{4}$ (E) $\frac{23}{2}$

Solution: (C) Let Σ be the sum. By the double angle identity of $\sin(2a) = 2\sin(a)\cos(a)$, $\Sigma = \dfrac{\sin^2(2°)}{4} + \dfrac{\sin^2(4°)}{4} + \dfrac{\sin^2(6°)}{4} + \cdots + \dfrac{\sin^2(88°)}{4} + \dfrac{\sin^2(90°)}{4}$.

$\sin(0°) = 0$, so $\Sigma = \dfrac{\sin^2(0°)}{4} + \dfrac{\sin^2(2°)}{4} + \dfrac{\sin^2(4°)}{4} + \dfrac{\sin^2(6°)}{4} + \cdots + \dfrac{\sin^2(88°)}{4} + \dfrac{\sin^2(90°)}{4}$.

By the identity $\sin^2(a) = \cos^2(90° - a)$, $\Sigma = \dfrac{\sin^2(0°)}{4} + \dfrac{\sin^2(2°)}{4} + \dfrac{\sin^2(4°)}{4} + \dfrac{\sin^2(6°)}{4} + \cdots + \dfrac{\sin^2(44°)}{4} + \dfrac{\cos^2(44°)}{4} + \cdots + \dfrac{\cos^2(2°)}{4} + \dfrac{\cos^2(0°)}{4}$.

Combining this with the Pythagorean identity of $\sin^2(a) + \cos^2(a) = 1$, $\Sigma = \dfrac{23}{4}$.

15. Find the product of all real solutions of the equation $(x^2+3x-4)^3+(2x^2-5x+3)^3=(3x^2-2x-1)^3$.
(A) -6 (B) -4 (C) $-\frac{1}{2}$ **(D)** 2 (E) $\frac{4}{3}$

> **Solution:** (D) Let $a=x^2+3x-4$ and $b=2x^2-5x+3$. Then $a+b=3x^2-2x-1$, so the given equation is equivalent to $a^3+b^3=(a+b)^3$, i.e. $3ab(a+b)=0$. Hence $a=0$ or $b=0$ or $a+b=0$. If $a=0$, then $x^2+3x-4=0$, and the solution of this equation are $x=1$ and $x=-4$. If $b=0$, then $2x^2-5x+3=0$, and the solution of this equation are $x=\frac{3}{2}$ and $x=1$. If $a+b=0$, then $3x^2-2x-1=0$, and the solution of this equation are $x=1$ and $x=-\frac{1}{3}$. Therefore the solutions of the given equation are $x=1$, $x=-4$, $x=\frac{3}{2}$, and $x=-\frac{1}{3}$, and their product is 2.

16. Determine all real values of a such that the equation $2a(x+1)^2-|x+1|+1=0$ has exactly four distinct real solutions.
(A) $a=\frac{1}{8}$ (B) $|a|<\frac{1}{8}$ (C) $a<\frac{1}{8}$ **(D)** $0<a<\frac{1}{8}$
(E) None of the answers (A) through (D) is correct.

> **Solution:** (D) Let $t=|x+1|$. The given equation has four distinct real solutions if and only if the equation $2at^2-t+1=0$ has two distinct real solutions. The last equation has two distinct real solutions if $1-8a>0$, i.e. $a<\frac{1}{8}$. The solutions of $at^2-t+1=0$ are $t_1=\frac{1-\sqrt{1-8a}}{4a}$ and $t_2=\frac{1+\sqrt{1-8a}}{4a}$. Since $t=|x+1|$, we have $t_1,t_2>0$. If $a<0$, then $t_2<0$, which contradicts $t_1,t_2>0$. Therefore, $0<a<\frac{1}{8}$.

17. Let $a_1,a_2,a_3,\ldots,a_{2013}$ be a geometric sequence with positive terms such that $a_1+a_2+\cdots+a_{2013}=2$ and $\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{2013}}=1$. Find the product $a_1a_2\cdots a_{2013}$.
(A) $2^{1006}\sqrt{2}$ (B) 2^{1006} (C) 2^{1007} (D) $2^{1007}\sqrt{2}$
(E) None of the answers (A) through (D) is correct.

> **Solution:** (A) Let q be the quotient of a geometric sequence that satisfies $a_1+a_2+\cdots+a_{2013}=2$ and $\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{2013}}=1$. Then $\frac{a_1(q^{2013}-1)}{q-1}=2$. If we divide the equation $\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{2013}}=1$ by q, we get $\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_{2013}}+\frac{1}{qa_{2013}}=\frac{1}{q}$. From the last two equations we get $a_1q^{2012}(q-1)=q^{2013}-1$, and from this equation and $\frac{a_1(q^{2013}-1)}{q-1}=2$, we get $a_{1007}=a_1q^{1006}=\sqrt{2}$. Since $a_1a_{2013}=a_2a_{2012}=\ldots=a_{1006}a_{1008}=a_{1007}^2=2$, we get $a_1a_2\cdots a_{2013}=(a_1a_{2013})(a_2a_{2012})\cdots(a_{1006}a_{1008})a_{1007}=2^{1006}\sqrt{2}$.
> Note: A geometric sequence with the given properties does not exist!

18. What is the remainder when $P(x)=1-2x+3x^2-4x^3+\cdots+99x^{98}-100x^{99}$ is divided by $Q(x)=x^2-1$?
(A) $2550x+2500$ **(B)** $-2550x+2500$ (C) $2550x-2500$ (D) $-2550x-2500$
(E) None of the answers (A) through (D) is correct.

> **Solution:** (B) Let $R(x)=ax+b$ be the reminder obtained when $P(x)$ is divided by $Q(x)$. Then $P(x)=S(x)Q(x)+R(x)$. Notice that $P(1)=-50$, $P(-1)=5050$, $Q(1)=Q(-1)=0$. Thus, from $P(1)=S(1)Q(1)+R(1)$ and $P(-1)=S(-1)Q(-1)+R(-1)$, we get $a+b=-50$ and $-a+b=5050$. Hence, $a=-2550$ and $b=2500$, and the reminder $R(x)$ is $-2550x+2500$.

19. Find the product of the solutions of the equation $\log_2(40 - 5x^2 + x^2 \cdot 2^x) = x + 3$.
(A) $-8\log_2 5$ (B) $\log_2 5^8$ (C) $\log_2 40$ (D) $-64\log_2 5$
(E) None of the answers (A) through (D) is correct.

Solution: (A) The given equation is equivalent to $40-5x^2+x^2\cdot 2^x = 2^{x+3}$, i.e. $(8-x^2)(5-2^x) = 0$. Thus, the roots of the equation are $\sqrt{8}$, $-\sqrt{8}$, and $\log_2 5$, and their product is $-8\log_2 5$.

20. The lengths of the sides of a triangle are consecutive integers and the largest angle is twice the smallest angle. What is the cosine of the smallest angle?
(A) $\frac{3}{4}$ (B) $\frac{7}{10}$ (C) $\frac{2}{3}$ (D) $\frac{9}{14}$ (E) None of the answers (A) through (D) is correct.

Solution: (A) Let θ be the smallest angle in the triangle and let n be the length of its shortest side. Using the law of sines we have $\frac{\sin\theta}{n} = \frac{\sin 2\theta}{n+2} = \frac{2\sin\theta\cos\theta}{n+2}$ which gives $\cos\theta = \frac{n+2}{2n}$. Using the law of cosines we get $n^2 = (n+1)^2 + (n+2)^2 - 2(n+1)(n+2)\cos\theta$, so $\cos\theta = \frac{-n^2+(n+1)^2+(n+2)^2}{2(n+1)(n+2)}$. Then $\frac{n+2}{2n} = \frac{-n^2+(n+1)^2+(n+2)^2}{2(n+1)(n+2)} = \frac{(n+1)(n+5)}{2(n+1)(n+2)} = \frac{n+5}{2(n+2)}$. So $n = 4$ and $\cos\theta = \frac{3}{4}$.

PART II: 10 INTEGER ANSWER PROBLEMS

1. Let x, y, z, and k be real numbers such that

$$\frac{7}{x+y} = \frac{k}{x+z} = \frac{11}{z-y}.$$

Find the value of k.

Solution: Answer: 18. If we substract the equation $7(z-y) = 11(x+y)$ from $7(x+z) = k(x+y)$, we get $7(x+y) = (k-11)(x+y)$. Since $x+y \neq 0$, we get $k - 11 = 7$, i.e. $k = 18$.

2. Let A be a set consisting of m ($m \neq 0$) consecutive integer numbers whose sum is $2m$ and let B be a set consisting of $2m$ consecutive integer numbers whose sum is m. Find the value of m if the difference between the largest element in B and the largest element in A is 99.

Solution: Answer: 201. Let $a+1, a+2, \ldots, a+m$ be the elements in A and $b+1, b+2, \ldots, b+2m$ be the elements in B. Then $ma + \frac{m(m+1)}{2} = 2m$ and $2mb + \frac{2m(2m+1)}{2} = m$. From the last two equations we get $a = \frac{3-m}{2}$ and $b = -m$. Since $(b+2m) - (a+m) = 99$, we get $m = 201$.

3. How many integer numbers between 10,000 and 99,999 are there such that all of their digits are distinct and the absolute value of the difference between the first and the last digit is 2?

Solution: Answer: 5040. From the set $\{0, 1, \ldots, 9\}$ there are sixteen pairs of numbers whose difference is ± 2 ($\{(0,2), (2,0), (1,3), (3,1), \ldots, (7,9), (9,7)\}$). Only the pair of digits $(0,2)$ can't be used as first digit 0 and last digit 2. For each of the 15 ordered pairs there are $8 \cdot 7 \cdot 6 = 336$ ways to fill the remaining middle three digits. Thus there are $15 \cdot 336 = 5040$ numbers of the required form.

4. Find the greatest common divisor of $2^{2015}+1$ and $2^{2013}+1$.

Solution: Answer: 3. Let d be the greatest common divisor of $2^{2015}+1$ and $2^{2013}+1$. Then d is a divisor of their difference $2^{2015}+1-2^{2013}-1=3\cdot 2^{2013}$. Since both numbers $2^{2015}+1$ and $2^{2013}+1$ are odd, it follows that $d=1$ or $d=3$. Since 3 divides both of them, it follows that $d=3$.

5. Let the product $12\cdot 15\cdot 16$, each factor written in base b, equal 3146 in base b. Let $s=12+15+16$, each term expressed in base b. What is s in base b?

Solution: Answer: 44. The product $12\cdot 15\cdot 16=3146$ (base b) can be written and $(b+2)(b+5)(b+6)=3b^3+b^2+4b+6$. Then $b^3-6b^2-24b-27=0$, i.e. $b=9$. Hence, $s=44$ in base b.

6. Let A, B, C, and D be the vertices of a regular tetrahedron, each of whose edges measures 1 meter. A bug, starting from vertex A, observes the following rule: at each vertex it chooses one of the three edges meeting at that vertex, each edge being equally likely to be chosen, and crawls along the edge to the vertex at its opposite end. Let $\frac{n}{2187}$ be the probability that the bug is at vertex A when it has crawled exactly 8 meters. Find the value of n.

Solution: Answer: 547. Let a_n be the probability that the bug is at vertex A after crawling exactly n meters, $n=0,1,2,\ldots$. Then $a_{n+1}=\frac{1}{3}(1-a_n)$ because the bug can be at vertex A after crawling $n+1$ meters if and only if it was not at A after crawling n meters (this has probability $1-a_n$) and from one of the other vertices it heads toward A (this is probability $\frac{1}{3}$). Since $a_0=1$, we have $a_1=0$, $a_2=\frac{1}{3}$, $a_3=\frac{2}{9}$, $a_4=\frac{7}{27}$, $a_5=\frac{20}{81}$, $a_6=\frac{61}{243}$, $a_7=\frac{182}{729}$, and $a_8=\frac{547}{2187}$.

7. Find the sum of all positive integer numbers n such that $n(n+16)$ is a square of an integer number.

Solution: Answer: 11. Since $n^2<n(n+16)<(n+8)^2$, we get that $n=2$ and $n=9$. Their sum is 11.

8. In a tournament each player played exactly one game against each of the other players. In each game the winner was awarded 1 point, the loser got 0 points, and each of the players earned $\frac{1}{2}$ point if the game was a tie. After the completion of the tournament, it was found that exactly half of the points earned by each player were earned in games against the ten players within the least number of points. What is the total number of players in the tournament?

Solution: Answer: 25. Let n be the number of players in the tournament. The total number of points earned at the tournament os equal to the total number of games played, which is $\frac{n(n-1)}{2}$. We will count the total number of points in another way. The ten lowest scoring players played exactly $\frac{10\cdot 9}{2}=45$ games between each other; however, this is only a half of the total points that the ten lowest scoring players earned at the tournament. The total number of points that the then lowest ranked players earned is 90. The players that were not ranked in the lowest ten played $\frac{(n-10)(n-11)}{2}$ games between each other, and therefore, they accumulated $\frac{(n-10)(n-11)}{2}$ points only when playing with each other. This is again a half of the total points earned by the players that were not ranked in the lowest ten. The other half of the points is from the games with the lowest ten players. So, the total number of points, counted differently, is $90+(n-10)(n-11)$. Therefore, we get the equation

$90+(n-10)(n-11)=\frac{n(n-1)}{2}$, and the solutions of this equation are $n=16$ and $n=25$. We discard the solution $n=16$: if they were only 16 players, then there would have been only 6 players that are not ranked in the 10 lowest, and the total number of points that these 6 players have earned would be 30, resulting in an average of 5 points for each of them. This is less that the average of 9 points gathered by the 10 lowest ranked players. Therefore, there were 25 players in the tournament.

9. Let $ABCD$ be a quadrilateral and let P be the intersection point of the diagonals AC and BD. Let $\angle ADB = 2\angle ACB$, $AD=BD=3$, and $BP=1$. What is the product of the length of the segments $\overline{AP}$ and $\overline{CP}$?

Solution: Answer: 5. Because we are asked to find the product of portions of line segments, we might be suspicious that one of the secant theorems is involved. But the secant theorems require a circle. Construct a circle with center at D and radius $AD=BD=3$. Let E be the intersection of the line BD with the circle. The point C must be on the circle because $\angle APB = 2\angle ACB$. Then, by the Internal Secant Theorem, $AP\cdot CP = EP\cdot PB = 5\cdot 1 = 5$.

10. Let a and b be relatively prime positive integer numbers such that $\frac{a}{b}$ is equal to the sum of all the real solutions of the equation $\sqrt[3]{3x-4}+\sqrt[3]{5x-6}=\sqrt[3]{x-2}+\sqrt[3]{7x-8}$. Find $a+b$.

Solution: Answer: 13. Let $a=\sqrt[3]{3x-4}$, $b=\sqrt[3]{5x-6}$, $c=\sqrt[3]{x-2}$, and $d=\sqrt[3]{7x-8}$. Then $a-c=d-b$ and $a^3-c^3=d^3-b^3$. From the equations $(a-c)^3=(d-b)^3$ and $a^3-c^3=d^3-b^3$ we get $ac(a-c)=db(d-b)$. This implies $ac=db$ or $a-c=d-b=0$; the second equation gives $x=1$. From $ac=db$ we get $\sqrt[3]{3x-4}\sqrt[3]{x-2}=\sqrt[3]{7x-8}\sqrt[3]{5x-6}$, which is equivalent to $4x^2-9x+5=0$. The solutions of the last quadratic equations are $x=1$ and $x=\frac{5}{4}$ and their sum is $\frac{9}{4}$. Hence, $a+b=13$.

The following problem, will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test.

TIE BREAKER PROBLEM

The numbers in the sequence $101, 104, 109, 116, \ldots$ are of the form $a_n = 100+n^2$ where $n=1,2,3,\ldots$. For each positive integer n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of the sequence $\{d_n\}$ as n ranges through the positive integers.

Solution: Answer: 401. If d_n divides $100+(n+1)^2$ and $100+n^2$, then it divides their difference, i.e. $d_n|(2n+1)$. Since $2(100+n^2)=n(2n+1)+(200-n)$, $d_n|(100+n^2)$, and $d_n|(2n+1)$, we get $d_n|(200-n)$. Hence, $d_n|((2n+1)+2(200-n))$, i.e. $d_n|401$. Thus, 401 is the largest possible value of d_n. If $n=200$, then $a_{200}=100\cdot 401$ and $a_{201}=101\cdot 401$. Therefore, the maximum value of the sequence $\{d_n\}$ is 401.