The Thirty-fourth Annual State High School Mathematics Contest

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NC STATE MATHEMATICS CONTEST
APRIL 2013

PART I: 20 MULTIPLE CHOICE PROBLEMS

1. Find the sum $1+2-3-4+5+6-7-8+\cdots+2009+2010-2011-2012+2013$.

   (A) 0 (B) 1 (C) 5 (D) -3 (E) None of the answers (A) through (D) is correct.

2. In $\triangle ABC$, $m(\angle ABC)=140^\circ$ and the points D and E lie on sides $\overline{AB}$ and $\overline{AC}$, respectively. If lengths AD, DE, EB, and BC are all equal, then the measure of $\angle BAC$ is

   (A) 5° (B) 6° (C) 7.5° (D) 8° (E) 10°

3. The numbers $a-b+2013$, $b-c+2013$, and $c-a+2013$ are three consecutive integer numbers. Find $a-3b+2c$.

   (A) -1 (B) 0 (C) 1 (D) -2 (E) 2

4. A circular disk is divided by $2n$ equally spaced radii ($n>0$) and one secant line. The maximum number of non-overlapping areas into which the disk can be divided is

   (A) $2n+1$ (B) $2n+2$ (C) $3n-1$ (D) $3n$ (E) $3n+1$

5. If $60^a=3$ and $60^b=5$, then $12^{\frac{1-a-b}{2(1-b)}}$ is

   (A) $\sqrt{3}$ (B) 2 (C) $\sqrt{5}$ (D) 3 (E) $\sqrt{12}$

6. Let x and y be positive integer numbers such that $\frac{1}{x}-\frac{1}{y}+\frac{1}{xy}=\frac{2}{5}$. Find $2x-3y$.

   (A) 3 (B) 23 (C) -11 (D) -2 (E) None of the answers (A) through (D) is correct.

7. If the sum of all angles except one in a convex polygon is 2190°, then the number of sides of the polygon is

   (A) 12 (B) 14 (C) 15 (D) 16 (E) 17

8. Let z and w be complex numbers such that $|z|=|w|=|z-w|$. Find $\left(\frac{z}{w}\right)^{2013}$.

   (A) -1 (B) 1 (C) $\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot i$ (D) $\frac{1}{2}-\frac{\sqrt{3}}{2}\cdot i$ (E) i

9. Let x_1 and x_2 be real roots of the equation $x^2-x+q=0$, where q is a real number. For which values of q, the expression $x_1^4+x_2^4$ has minimum value?

   (A) $\frac{1}{4}$ (B) 1 (C) 0 (D) -1 (E) None of the answers (A) through (D) is correct.

10. Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by $f(x)=ax^5+bx^3+cx-7$ where a, b, and c are real numbers. If $f(-5)=5$, find $f(5)$.

    (A) -5 (B) 10 (C) 5 (D) 12 (E) None of the answers (A) through (D) is correct.

11. A drawer contains 10 socks, 6 red, 4 blue. If 4 socks are pulled from the drawer at random, what is the probability that there are more red socks than blue socks?

(A) $\frac{5}{21}$ (B) $\frac{19}{42}$ (C) $\frac{1}{2}$ (D) $\frac{3}{5}$ (E) $\frac{31}{42}$

12. Let

$$U(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$$

For how many distinct positive integers, n, is $U(U(U(U(U(n))))) = 1$?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 8

13. Two candles of the same length are made of different materials so that one burns out completely at a uniform rate in 3 hours and the other, in 4 hours. At what time P.M. should the candles be lightened so that, at 4:00 P.M., one stub is twice the length of the other?

(A) 1:24 (B) 1:28 (C) 1:36 (D) 1:40 (E) 1:48

14. Evaluate the sum

$$\sin^2(1^\circ)\cos^2(1^\circ) + \sin^2(2^\circ)\cos^2(2^\circ) + \sin^2(3^\circ)\cos^2(3^\circ) + \ldots + \sin^2(44^\circ)\cos^2(44^\circ) + \sin^2(45^\circ)\cos^2(45^\circ)$$

(A) $\frac{23}{8}$ (B) $\frac{45}{8}$ (C) $\frac{23}{4}$ (D) $\frac{45}{4}$ (E) $\frac{23}{2}$

15. Find the product of all real solutions of the equation $(x^2+3x-4)^3 + (2x^2-5x+3)^3 = (3x^2-2x-1)^3$.

(A) -6 (B) -4 (C) $-\frac{1}{2}$ (D) 2 (E) $\frac{4}{3}$

16. Determine all real values of a such that the equation $2a(x+1)^2 - |x+1| + 1 = 0$ has exactly four distinct real solutions.

(A) $a = \frac{1}{8}$ (B) $|a| < \frac{1}{8}$ (C) $a < \frac{1}{8}$ (D) $0 < a < \frac{1}{8}$
(E) None of the answers (A) through (D) is correct.

17. Let $a_1, a_2, a_3, \ldots, a_{2013}$ be a geometric sequence with positive terms such that $a_1 + a_2 + \cdots + a_{2013} = 2$ and $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2013}} = 1$. Find the product $a_1 a_2 \cdots a_{2013}$.

(A) $2^{1006}\sqrt{2}$ (B) 2^{1006} (C) 2^{1007} (D) $2^{1007}\sqrt{2}$
(E) None of the answers (A) through (D) is correct.

18. What is the remainder when $P(x) = 1 - 2x + 3x^2 - 4x^3 + \cdots + 99x^{98} - 100x^{99}$ is divided by $Q(x) = x^2 - 1$?

(A) $2550x + 2500$ (B) $-2550x + 2500$ (C) $2550x - 2500$ (D) $-2550x - 2500$
(E) None of the answers (A) through (D) is correct.

19. Find the product of the solutions of the equation $\log_2(40 - 5x^2 + x^2 \cdot 2^x) = x + 3$.

(A) $-8\log_2 5$ (B) $\log_2 5^8$ (C) $\log_2 40$ (D) $-64\log_2 5$
(E) None of the answers (A) through (D) is correct.

20. The lengths of the sides of a triangle are consecutive integers and the largest angle is twice the smallest angle. What is the cosine of the smallest angle?

(A) $\frac{3}{4}$ (B) $\frac{7}{10}$ (C) $\frac{2}{3}$ (D) $\frac{9}{14}$ (E) None of the answers (A) through (D) is correct.

PART II: 10 INTEGER ANSWER PROBLEMS

1. Let x, y, z, and k be real numbers such that
$$\frac{7}{x+y} = \frac{k}{x+z} = \frac{11}{z-y}.$$
Find the value of k.

2. Let A be a set consisting of m ($m \neq 0$) consecutive integer numbers whose sum is $2m$ and let B be a set consisting of $2m$ consecutive integer numbers whose sum is m. Find the value of m if the difference between the largest element in B and the largest element in A is 99.

3. How many integer numbers between 10,000 and 99,999 are there such that all of their digits are distinct and the absolute value of the difference between the first and the last digit is 2?

4. Find the greatest common divisor of $2^{2015} + 1$ and $2^{2013} + 1$.

5. Let the product $12 \cdot 15 \cdot 16$, each factor written in base b, equal 3146 in base b. Let $s = 12 + 15 + 16$, each term expressed in base b. What is s in base b?

6. Let A, B, C, and D be the vertices of a regular tetrahedron, each of whose edges measures 1 meter. A bug, starting from vertex A, observes the following rule: at each vertex it chooses one of the three edges meeting at that vertex, each edge being equally likely to be chosen, and crawls along the edge to the vertex at its opposite end. Let $\frac{n}{2187}$ be the probability that the bug is at vertex A when it has crawled exactly 8 meters. Find the value of n.

7. Find the sum of all positive integer numbers n such that $n(n+16)$ is a square of an integer number.

8. In a tournament each player played exactly one game against each of the other players. In each game the winner was awarded 1 point, the loser got 0 points, and each of the players earned $\frac{1}{2}$ point if the game was a tie. After the completion of the tournament, it was found that exactly half of the points earned by each player were earned in games against the ten players within the least number of points. What is the total number of players in the tournament?

9. Let $ABCD$ be a quadrilateral and let P be the intersection point of the diagonals AC and BD. Let $\angle ADB = 2\angle ACB$, $AD = BD = 3$, and $BP = 1$. What is the product of the length of the segments $\overline{AP}$ and $\overline{CP}$?

10. Let a and b be relatively prime positive integer numbers such that $\frac{a}{b}$ is equal to the sum of all the real solutions of the equation $\sqrt[3]{3x-4} + \sqrt[3]{5x-6} = \sqrt[3]{x-2} + \sqrt[3]{7x-8}$. Find $a + b$.

The following problem, will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test.

TIE BREAKER PROBLEM

The numbers in the sequence $101, 104, 109, 116, \ldots$ are of the form $a_n = 100 + n^2$ where $n = 1, 2, 3, \ldots$. For each positive integer n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of the sequence $\{d_n\}$ as n ranges through the positive integers.