State Mathematics Finals: Geometry

April 26, 2012

1. What is the approximate diameter of a wheel that rolled for 314 meters by turning 200 times?

   x = diameter $\Longrightarrow$ $200 \cdot x \cdot \pi = 314$ $\Longrightarrow$ $x \approx 314 / (200 \cdot 3.14) = 0.5$ meters.

   a) 25 cm **b) 50 cm** c) 100 cm d) 50π cm e) none of these

2. What is the area of a rectangle with a side of length 12 cm and a diagonal of length 15 cm?

   h = height $\Longrightarrow$ $h^2 = 15^2 - 12^2 = 81$ $\Longrightarrow$ Area = $9 \cdot 12$ cm^2.

   a) 108 cm^2 b) 54 cm^2 c) 135 cm^2 d) 180 cm^2 e) 72 cm^2

3. Visualize a quadrilateral with at least one right angle whose vertices lie on a circle. Two non-adjacent vertices are 22 cm apart, and the other two are 30 cm apart. What is the area of the circle?

   Since one angle is 90° two opposing vertices are on the diagonal. $\Longrightarrow$ Diagonal = 30 cm.

   a) 52π cm^2 b) 121π cm^2 c) 169π cm^2 **d) 225π cm^2** e) 330π cm^2

4. Seven sailors, four Russians and three Americans, arrived in a submarine. If they emerged from the vessel in random order, what is the probability that the order was: A, R, R, A, A, R, R? ("A" represents an American sailor, and "R" a Russian one.)

   Probability of (A,R,R,A,A,R,R) = (3/7)(4/6)(3/5)(2/4)(1/3)(2/2)(1/1)

   a) 1/21 b) 1/32 **c) 1/35** d) 1/42 e) 1/128

5. A triangle and a square went into a bar, and it did not take long for them to get into an argument about who was bigger. "I am bigger because I am two inches taller!" said the triangle. To which the square replied, "No, I am bigger because my area is two square inches larger!" "No!" interjected the bartender, "You are the same, because you both have the same width." He then threw them out the door. Assuming all told the truth, how wide were the two polygons?

   x = width $\Longrightarrow$ $0.5 \cdot (x+2) \cdot x = x^2 - 2$ $\Longrightarrow$ $x^2 - 2x - 4 = 0$

   a) 2 in. b) 2.5 in. c) $1+\sqrt{2}$ in. d) $2+\sqrt{2}$ in. **e) $1+\sqrt{5}$ in.**

6. What is the height of a right triangle, whose hypotenuse is the base, and whose area and perimeter are both 30?

Let a, b & c be the sides. $\Longrightarrow$ $a+b=30-c$ & $ab=60$ $\Longrightarrow$ $(a+b)^2 = a^2+2ab+b^2 = c^2+120 = (30-c)^2$
Thus $120 = 900 - 60c$ $\Longrightarrow$ $c = 13$ $\Longrightarrow$ Area $= 0.5 \cdot h \cdot 13 = 30$ $\Longrightarrow$ 60/13

a) $2\frac{4}{13}$ **b) $4\frac{8}{13}$** c) $5\frac{12}{13}$ d) 6 e) $12\frac{5}{13}$

7. To get from point A to point B a taxi cab driver had to drive 4 kilometers to the North, 4 kilometers to the West, and then 7 kilometers on a road heading southwest. If a crow flew directly from point A to point B how far would it fly?

First two legs distance = $\sqrt{(4^2+4^2)} = \sqrt{32}$; Total distance = $\sqrt{(32+7^2)} = \sqrt{81}$

a) 6 km b) 9 km c) 11 km d) $(7+4\sqrt{2})$ km e) 15 km

8. A rectangle with a perimeter of 52 inches doubles in area if 4 inches are added to both its width and length. What is the area of the original rectangle?

x = width $\Longrightarrow$ $2x \cdot (26-x) = (x+4) \cdot (30-x)$ $\Longrightarrow$ $x^2 - 26x + 120 = 0$ $\Longrightarrow$ $x = 20, 6$

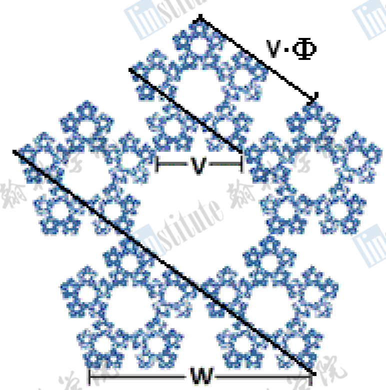
a) 120 in^2 b) 136 in^2 c) 169 in^2 d) 180 in^2 e) none of these

9. How many different combinations of distinct single digit natural numbers {1, 2, …, 9} can you find so that their sum is 13?

{4,9},{5,8},{6,7}, {1,3,9},{1,4,8},{1,5,7},{2,3,8},{2,4,7},{2,5,6},{3,4,6}, {1,2,3,7},{1,2,4,6},{1,3,4,5}

a) 10 b) 11 c) 12 **d) 13** e) 14

10. Consider the fractal image that is made up of 5 self-similar parts as shown. If each of the parts shares exactly one point with each of its neighbors by what factor is the width, W, of the whole image wider than the width, V, of one of the five self-similar parts?



a) $\sqrt{5}$ b) $\frac{\sqrt{5}+2}{2}$ **c) $\frac{\sqrt{5}+3}{2}$**

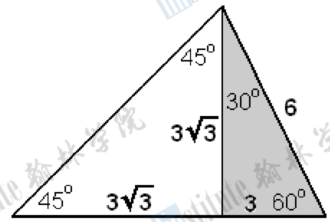
W=V·Φ+V & W·Φ=2·V·Φ+V $\Longrightarrow$ V·(Φ + 1)·Φ = V·(2Φ +1)
$\Longrightarrow$ $\Phi^2 - \Phi - 1 = 0$ $\Longrightarrow$ $\Phi = (1+\sqrt{5})/2$. W = V·Φ+V $\Longrightarrow$ W/V = Φ +1

11. If you roll three dice, what is the probability the total value on two of them will add to seven?

Pr(Dice A&B add to 7) = 6/36   Pr(Dice A&B a pair, Dice A&C add to 7) = (1/6)(1/6) = 1/36
Pr (Above two cases not occurring, Die C plus Die A or B add to 7) = (4/6)(2/6) = 8/36
$\Longrightarrow$ 6/36 + 1/36 + 8/36 = 15/36

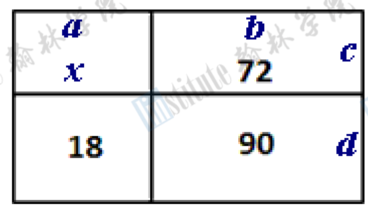
a) $\frac{1}{6}$ b) $\frac{11}{36}$ **c) $\frac{5}{12}$**   d) $\frac{5}{72}$ e) $\frac{17}{72}$

12. Consider a triangle with two angles measuring 45° and 60°. The shortest side of this triangle has length 6, what is the length of its longest side?
Divide triangle into two right triangles and solve.



a) $3\cdot(1+\sqrt{3})$ b) $3\sqrt{6}$ c) 10
d) 12   e) none of these

13. A rectangle is partitioned into four smaller rectangular pieces, three of which have areas, 18, 72 and 90. The fourth one is not given. What is the area of the missing piece?



$(ac)(bd) = (ad)(bc) \Longrightarrow x\cdot 90 = 18\cdot 72 \Longrightarrow x = 18\cdot 72/90$

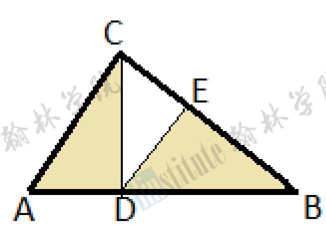
a) 12   b) 13.6   c) 14   **d) 14.4**

14. The internal angles of a pentagon have measures, x, x + 5°, x + 10°, x + 20° and x + 40°. What is the measure of the smallest angle?

$x + (x+5°) + (x+10°) + (x+20°) + (x+40°) = 540° \Longrightarrow 5x + 75° = 540°$

a) 88°   **b) 93°**   c) 103°   d) 108°   e) none of these

15. A right triangle, ΔABC, is partitioned into three similar triangles such that the larger two are congruent (i.e. ΔACD≅ ΔDBE).
If the shorter leg of ΔABC has length 2, what is the length of ΔABC's hypotenuse?



Let AB = x. AD : AC = AC : AB $\Longrightarrow$ AD:2 = 2:x $\Longrightarrow$ AD = 4/x
AB = AD+DB $\Longrightarrow$ $x = 4/x + 2 \Longrightarrow x^2 - 2x - 4 = 0$

a) 3   b) $5-\sqrt{5}$ **c) $1+\sqrt{5}$**   d) $3+\sqrt{2}$ e) none of these

16. A standard die has 6 sides numbered 1 through 6. The numbers on opposite sides always add to 7. When you roll a standard die it is possible to see three sides at one time. What is the probability that the sum of the numbers of the three visible sides is 13?

Eight possible outcomes: {4,5,6}{3,5,6}{4,2,6}{3,2,6}…. None work.

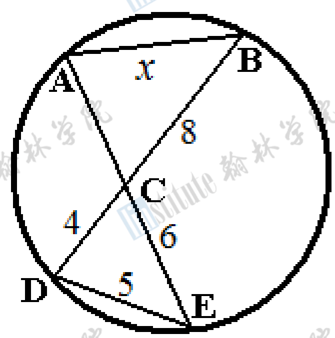
a) $\frac{1}{6}$ b) $\frac{1}{8}$ c) $\frac{1}{12}$ d) $\frac{1}{36}$ **e) 0 (i.e. it's impossible)**

17. A circle centered at the origin of radius 10 is intersected by the line $3y - x = 10$ at two points. What is the distance between the two points?

$x^2+y^2=100$ & $3y-10=x \implies (3y-10)^2 + y^2=100 \implies 10y^2 - 60y = 0 \implies$ Points: $(-10,0)$ & $(8,6)$

a) 10 **b) $6\sqrt{10}$** c) $12\sqrt{5}$ d) $10\sqrt{6}$ e) 24

18. Given a circle with four chords, two of which intersect at *C*. If $BC = 8$, $CD = 4$, $DE = 5$, and $EC = 6$, what is *x*, the length of $\overline{AB}$?



$x : 8 = 5 : 6 \implies$ **$x = 40/6$**

a) 10 b) 9 c) 9.6 d) 6 **e) none of these**

19. A rectangular solid with a black surface area and dimensions 10×8×6 is cut into unit cubes. Assuming the solid's interior is not black, what fraction of these cubes has no black side?

Total number of unit cubes / Internal unit cubes = (8×6×4)/ (10×8×6) = 4/10

a) $\frac{13}{60}$ b) $\frac{5}{12}$ c) $\frac{5}{8}$ **d) $\frac{2}{5}$** e) $\frac{3}{5}$

20. Let $n = k^2 - k + 2$ where *k* is a natural number. Which of the following statements are true for all values of *k*?

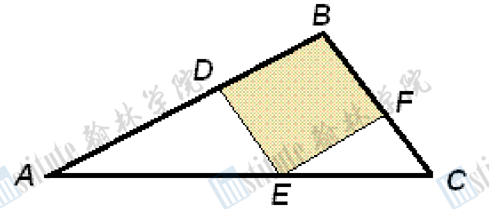
i. *n* is even ii. *n* is never divisible by 3 iii *n* is never divisible by 5

Check the first 5 whole numbers for *k* since problem addresses numbers of mod 2, 3, and 5.

$k=0 \implies n=2$; $k=1 \implies n=2$; $k=2 \implies n=4$; $k=3 \implies n=8$; $k=4 \implies n=14$;

a) i only b) i and ii c) i and iii **d) all of them** e) none of them

21. In $\Delta\, ABC$ $AB : BC = 2 : 1$, points D, E, and F are on $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$, as shown, and the quadrilateral $BDEF$ is a rhombus. Find the ratio of the areas of quadrilateral $BDEF$ to ΔABC.



$\Delta ABC \sim \Delta ADE \sim \Delta EFC \implies AB/BC = AD/DE = EF/FC = 2$. Let $x = FC \implies EF = 2x \implies AD = 4x$
$EF = BF \implies BC = 3x$. Let $R =$ Area $\Delta EFC \implies$ Area $\Delta ABC = 9\cdot R$, Area $BDEF = 9\cdot R - 4\cdot R - R$.

a) 4:9 b) $2:\sqrt{3}$ c) $2:3$ d) 2:1 e) none of these

22. The volume of a large spherical balloon is doubled. By what factor is the surface area of the balloon increased?

Let r = radius. Volume $\propto r^3$ and Surface $\propto r^2$ $\implies$ Surface $\propto$ (Volume)$^{2/3}$

a) 8 b) $\sqrt{2}$ c) $2\sqrt{2}$ **d) $\sqrt[3]{4}$** e) none of these

23. The distance from a vertex to the orthocenter of an acute triangle is the same as the distance [and direction] from that vertex to the circumcenter (outer center) of the acute triangle. Determine the measure of the interior angle at this vertex.

Criterion true for equilateral triangle $\implies$ angle = 60°

a) 15° b) 30° c) 45° **d) 60°** e) 75°

24. Consider trapezoids with sides of length 1, 4, 4, and 5. Find the sum of the two diagonals of the trapezoid with the smallest area.

Trapezoid with smaller area has parallel sides of length 1 & 4 and height 4.
Lengths of diagonals are $\sqrt{(4^2+1^2)}$ and $\sqrt{(4^2+4^2)}$.

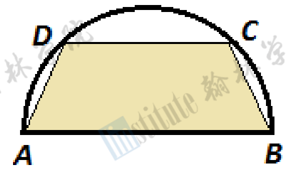
a) 6 b) $4\sqrt{3}$ c) $2\sqrt{21}$ **d) $\sqrt{17}+4\sqrt{2}$** e) none of these

25. Find the area on xy-plane determined by the inequality, $|2x-6|+|y-2| \le 6$.

Vertices of quadrilateral are {(3, 8), (3, –4), (0, 2), (6, 2)} Area = ½(6 × 12)

a) 12 **b) 36** c) 42 d) 56 e) 72

26. A trapezoid $ABCD$ is inscribed in a semi-circle of diameter 8 inches as shown. If $\overline{CD}$ is parallel to $\overline{AB}$ and the perimeter of the trapezoid $ABCD$ is 20 inches, what is the area of this trapezoid?



Vertices on circle $x^2 + y^2 = 4^2$, $DC = 2x$, y=height. $AB = 8$.
Perimeter = $20 = 8 + 2x + 2\sqrt{(4-x)^2 + y^2} \implies 6-x = \sqrt{16-8x+x^2+y^2} \implies (6-x)^2 = 32-8x$
$\implies x^2 - 4x + 4 = 0 \implies x = 2 \implies y = \sqrt{12} \implies$ Area $= 6\sqrt{12}$

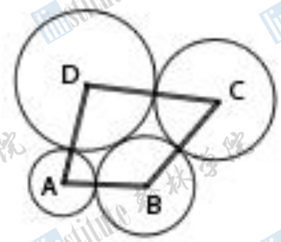
a) $12\sqrt{3}$ in^2 b) $12\sqrt{5}$ in^2 c) 16 in^2 d) 20 in^2 e) 32 in^2

27. A line on the xy-plane with a negative slope passes point, (2,1). The line crosses the x-axis at A and the y-axis at B. What is the [minimum] area of ΔAOB if O is the origin, (0, 0)?

Area of $\Delta AOB = \frac{1}{2}AB = R$; Line: $y = (x-A)/(2-A) \implies B = -A/(2-A)$ or $R = \frac{1}{2}A^2/(A-2)$
$\implies 2R(A-2) = A^2 \implies A^2 - 2RA + 4R = 0 \implies A = R \pm\sqrt{R^2 - 4R} \implies R \geq 4$.

a) 1.5 b) 2 **c) 4** d) 5 e) 5.5

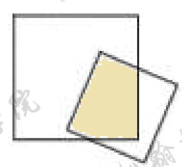
28. Quadrilateral $ABCD$ has sides that measure 7, 12, 15, and 10. Four circles, each centered at one of the quadrilateral's vertices, are mutually tangent as shown. Find [a possible] sum of the areas of the four circles.



Radii of circles A, B,C & D are x, $(7-x)$, $(5+x)$ & $(10-x)$ respectively.
Area = $\pi(x^2+(7-x)^2+(5+x)^2+(10-x)^2 = 4x^2-24x+174 = 4\pi(x-3)^2 + 142\pi$

a) 86π b) 121π c) 129π **d) 142π** e) 484π

29. Square A and square B overlap as seen in the picture. The overlapped area is $\frac{1}{4}$ of the square A and $\frac{2}{3}$ of the square B. Find the ratio of the perimeter of square A to that of square B.



Let area A = a^2 and area B = $b^2 \implies (1/4)\,a^2 = (2/3)\,b^2 \implies a^2/b^2 = 8/3 \implies a/b = \sqrt{8}/\sqrt{3}$

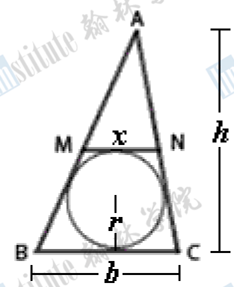
a) 3:8 b) $2:\sqrt{3}$ **c) $2\sqrt{2}:\sqrt{3}$** d) 8:3 e) none of these

30. Consider a right triangle ΔABC whose legs $\overline{AC}$ and $\overline{BC}$ measure 1 and 2, respectively. Points P, N, and M are on $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$, respectively, so that $\overline{PN}$ and $\overline{PM}$ are perpendicular. Find the minimum possible length of $\overline{MN}$.

Let vertices A, B & C be the points (1, 0), (0, 2) and (0, 0) respectively on the Cartesian plane.
Point P is on the line $x + 2y = 2$. Let $(2–2p, p)$ be the coordinates of point P.
Segments parallel $\Longrightarrow$ $N = (2–2p, 0)$ and $M = (0, p)$ $\Longrightarrow (MN)^2 = (2 – 2p)^2 + p^2$
$(MN)^2 = 5p^2 – 8p + 4 = 5(p^2 – 8/5\, p + 4/5) = 5((p–4/5)^2 + 4/25) \Longrightarrow (MN)^2 = 5(4/25) = 4/5$ is Min.

a) $\frac{\sqrt{5}}{2}$ **<u>b)</u>** $\frac{2\sqrt{5}}{5}$ c) $\frac{4\sqrt{5}}{5}$ d) $\sqrt{5}$ e) none of these

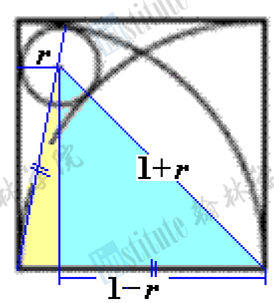
31. A circle is inscribed in a triangle ABC with a perimeter of 10. Point M is on $\overline{AB}$, Point N is on $\overline{AC}$, and $\overline{MN}$ is parallel to $\overline{BC}$ and tangent to the circle as shown. Find the maximum of $\overline{MN}$.



Area or $\Delta ABC = ½\, r\cdot$perimeter $= 5r$; or area $= ½bh. \Longrightarrow 5r = ½bh \Longrightarrow r/h = b/10$
Also $x/b = (h–2r)/h \Longrightarrow x/b = 1 – 2(r/h) \Longrightarrow x = b(1 – 2(b/10)) = b – b^2/5$.
maximum of $b – b^2/5$ is when $b = 5/2$.

a) $\frac{2}{3}$ b) $\frac{3}{2}$ c) $\frac{4}{5}$ **<u>d)</u>** $\frac{5}{4}$ e) none of these

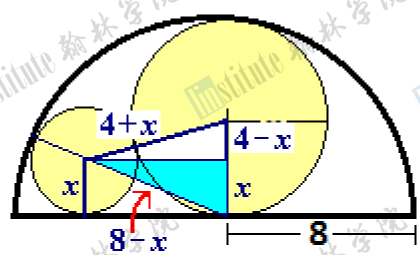
32. Consider a unit square containing two arcs each with radius 1 as shown in the picture. Find the radius of the circle that is tangent to the two arcs and the left side of the square.



$$(1–r)^2 – r^2 = (1+r)^2 – (1–r)^2 \Longrightarrow –2r + 1 = 4r$$

<u>a)</u> $\frac{1}{6}$ b) $\frac{1}{3}$ c) $\frac{5}{12}$ d) $\frac{2}{5}$ e) $\frac{1}{2}$

33. Two tangent circles are inscribed in a semi-circle with a radius of 8 as shown. If the larger circle has a radius of 4, what is the radius of the smaller circle?



$$(8–x)^2 – x^2 = (4+x)^2 – (4–x)^2 \Longrightarrow 64 – 16x = 16x$$

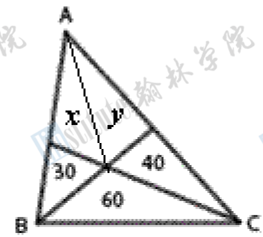
a) $\sqrt{3}$ **<u>b) 2</u>** c) $\frac{17}{8}$ d) $\sqrt{8}$ e) 3

34. What is the probability that a triangle using three vertices of a regular hexagon is an isosceles triangle?

Total number of possible triangles is 6 choose 3 = 20. There are two possible equilateral triangles & 6 possible (non-equilateral) isosceles triangles. Probability = 8 / 20.

a) $\frac{1}{6}$ b) $\frac{1}{3}$ c) $\frac{5}{12}$ **<u>d) $\frac{2}{5}$</u>**   e) $\frac{1}{2}$

35. The areas of three small triangles are 40, 60, and 30 as seen in the picture. What is the area of triangle *ABC*?



$x / 30 = (x+y+40) / 90$ and $y / 40 = (x+y+30) / 100$

$3x = x+y+40$ or $y = 2x - 40$, also $2.5\, y = x+y+30 \implies 1.5(2x-40) = x+30$

$\implies x = 45$ & $y = 50$

a) 190   b) 200   **<u>c) 225</u>**   d) 240   e) 255

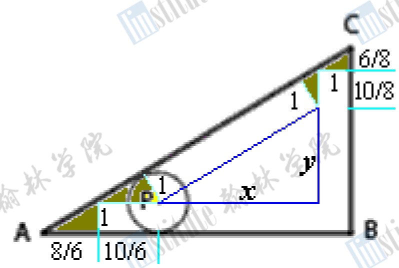
36. Point *P* divides a line segment $\overline{AB}$ such that $AP : PB = 2:3$, and point *Q* divides the line segment such that $AQ : QB = 3:4$. If $PQ = 2$, find *AB*.

$AP{:}PB = 2{:}3 \implies AP/AB = 2/5$ & $AQ{:}QB = 3{:}4 \implies QB/AB = 4/7$ &

$AB = AP+PQ+QB = AB(2/5)+2+AB(4/7) \implies AB(1 - 2/5 - 4/7) = 2 \implies AB(1/35) = 2$

a) 35   b) 40   c) 50   d) 60   **<u>e) 70</u>**

37. Consider a circle with a radius of 1 in a triangle which has sides 6, 8, and 10. If the circle rolled (like a wheel) along the edge of all the sides of ΔABC once (until it comes back to the starting position) as shown in the picture, what is the distance that the center P travelled?

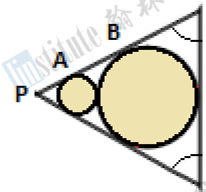


Let *x* and *y* be the lengths of legs of internal triangle. $x = 8 - 1 - 18/6 = 4$, $y = 6 - 1 - 16/8 = 3$.

Internal triangle ~ $\Delta ABC \implies$ Hypotenuse = 5.

a) 10   **<u>b) 12</u>**   c) 14   d) 18   e) 24

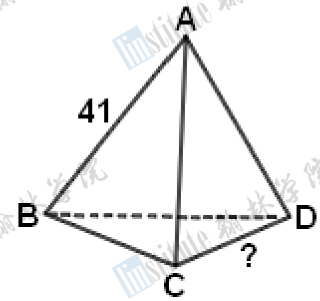
38. Two tangent circles are inscribed in an isosceles triangle as shown in the picture. If $PA = 2$ and $AB = 4$, what is the area of the smaller circle?



Let C and D be centers of the two circles, and r radius of smaller circle.
ΔPAC and ΔPBD are right triangles with $\Delta PAC \sim \Delta PBD$. $PB = PA+AB$
$PB = 6 \Longrightarrow$ radius of larger circle $= 3r$ and $CD = 4r$. Also $CD = 2\sqrt{(2^2+r^2)}$.
$\Longrightarrow 16r^2 = 4(4+r^2) \Longrightarrow 3r^2 = 4$

a) $\frac{2}{3}\pi$ b) π **c) $\frac{4}{3}\pi$** d) 4π e) none of these

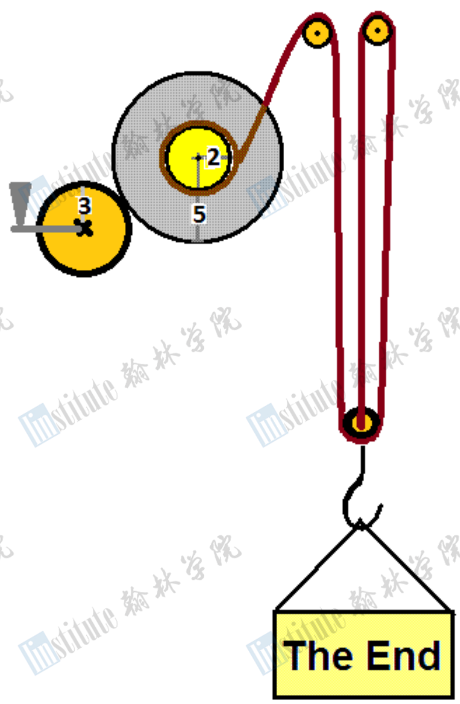
39. Consider a tetrahedron with edges of length 7, 13, 18, 27, 36, and 41. If $AB = 41$ find CD.



If triangle has sides of length $a \le b \le c$ then $a+b>c$. Thus these pairs of triples are feasible triangle lengths {7 or 13, 36, 41}{18, 27, 41}.
CD ≠ 7 since 7 cannot form two triangles without using 41 & 36.

a) 7 **b) 13** c) 18 d) 27 e) 36

40. A pulley constructed as shown in the figure on the right is designed to lift and lower a sign. One end of the rope is fixed to the hook while the other is on a spool with a 2 inch radius. The spool is attached to a 5 inch wheel, which is turned by a crank mounted on a 3 inch disk. If you were to turn the crank counterclockwise for 3 turns, how much will the sign move?



Turing crank 3 turns counter clockwise rotates 5-inch wheel $3 \times (3/5)$ turns clockwise wrapping up rope.

1.8 turns will wrap roughly $1.8 \times 4 \times 3.14$ inches of rope.

This will raise sign $1.8 \times 4 \times 3.14/3$ or roughly 7.5 inches.

Ans: b) Between 6 and 8 inches up