Solutions

PART I: 20 MULTIPLE CHOICE PROBLEMS

(1) Answer: (d). For any real number y we have $3 \circ y = 4 \cdot 3 - 3y + 3y = 12$. So, $3 \circ y = 12$ holds for every real number y.

(2) Answer: (a). $100^2 - 99^2 + 98^2 - 97^2 + \cdots + 2^2 - 1^2 = (100-99)(100+99) + (98-97)(98+97) + \cdots + (2-1)(2+1) = 100 + 99 + 98 + 97 + \cdots + 2 + 1 = \frac{100 \cdot 101}{2} = 5050$.

(3) Answer: (b). Since $\frac{19n+7}{7n+11} = \frac{1}{7}\left(19 - \frac{160}{7n+11}\right)$, we have that $7n+11|160$ and $7\Big|\left(19 - \frac{160}{7n+11}\right)$. Hence $n \in \{3, -1, -3, -13\}$ and their sum is -14.

(4) Answer: (c). Let the length of the legs are $3a$ and $3b$. Applying the Pythagorean Theorem twice, we get $4a^2 + b^2 = \sin^2 x$ and $a^2 + 4b^2 = \cos^2 x$. By adding these two equations, we get $5(a^2 + b^2) = 1$. Hence, the hypothenuse has length $\sqrt{9a^2 + 9b^2} = 3\frac{\sqrt{5}}{5}$.

(5) Answer: (c). Since $2025 = 3^4 \cdot 5^2$, it follows that 2025 has 15 divisors and 10 of them are divisible by 5. Hence, the probability that a randomly chosen divisor of 2025 is divisible by 5 is $\frac{10}{15} = \frac{2}{3}$.

(6) Answer: (d). $a + b + ab = 2012$ is equivalent to $(a+1)(b+1) = 2013 = 3 \cdot 11 \cdot 61$. We have the following systems of equations: $a+1 = 1, b+1 = 2013$ (which does not have a solution in the set of the positive integers); $a+1 = 3, b+1 = 671$; $a+1 = 11, b+1 = 183$; $a+1 = 33, b+1 = 61$; $a+1 = 61, b+1 = 33$; $a+1 = 183, b+1 = 11$; $a+1 = 671, b+1 = 3$, and $a+1 = 2013, b+1 = 3$ (which also does not have a solution in the set of the positive integers).

(7) Answer: (e). $z_0 = 1 - i$, $z_1 = \frac{1+2i}{5}$, $z_2 = -2 + i$, $z_3 = z_0 = 1 - i$. Hence $z_{2012} = z_{2010+2} = z_2 = -2 + i$.

(8) Answer: (b). The distance between the center of the sphere of radius 2 and the plane determined by the centers of the spheres of radius 1 is $\sqrt{3^2 - \left(\frac{2\sqrt{3}}{3}\right)^2} = \frac{\sqrt{69}}{3}$. Therefore, the plane to the top of the larger sphere is $3 + \frac{\sqrt{69}}{3}$.

(9) Answer: (c). Since $n! + (n+1)! = n!(n+2)$, we have $1! \cdot 3 - 2! \cdot 4 + 3! \cdot 5 - 4! \cdot 6 + \cdots - 2012! \cdot 2014 + 2013! = 1! + 2! - 2! - 3! + 3! + 4! - 4! - 5! + \cdots - 2012! - 2013! + 2013! = 1$.

(10) Answer: (a). Let $n \geq 2$. Then $f(1) + f(2) + \cdots + f(n) = n^2 f(n)$ and $f(1) + f(2) + \cdots + f(n-1) = (n-1)^2 f(n-1)$ which implies $f(n) = \frac{n-1}{n+1} \cdot f(n-1)$. Multiplying the equations: $f(2) = \frac{1}{3} \cdot f(1)$, $f(3) = \frac{2}{4} \cdot f(2)$, $f(4) = \frac{3}{5} \cdot f(2)$, ..., $f(n-1) = \frac{n-2}{n} \cdot f(n-2)$, $f(n) = \frac{n-1}{n+1} \cdot f(n-1)$, we get $f(n) = \frac{2(n-1)!}{(n+1)!} \cdot f(1) = \frac{2020}{2012 \cdot 2013} = \frac{505}{1012539}$.

(11) Answer: (d). If Andrew is the overall winner, then he needs to win the seventh game, and in the first six he must win four and loose two games. The probability that this will happen is $P_A = \binom{6}{4} \cdot \left(\frac{2}{3}\right)^4 \cdot \left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right) = \frac{160}{729}$. Similarly, the probability that Jordan will win is $P_J = \binom{6}{4} \cdot \left(\frac{1}{3}\right)^4 \cdot \left(\frac{2}{3}\right)^2 \cdot \left(\frac{1}{3}\right) = \frac{20}{729}$. Therefore, the probability that the overall winner is decided in exactly seven games is $\frac{160}{729} + \frac{20}{729} = \frac{20}{81}$.

(12) Answer: (c). Let d be the difference of the arithmetic sequence. From $\frac{S_m}{S_n} = \frac{m^2}{n^2}$ we get $\frac{2a_1+(m-1)d}{a2_1+(n-1)d} = \frac{m}{n}$, which implies $2a_1(n-m) = (n-m)d$. If $m \neq n$, we get $d = 2a_1$, which implies $\frac{a_m}{a_n} = \frac{2m-1}{2n-1}$. If $m = n$, $\frac{a_m}{a_n} = \frac{2m-1}{2n-1}$ is clearly satisfied.

(13) Answer: (a). From $9\sin^2 A + 16\cos^2 B + 24\sin A\cos B = 36$ and $16\sin^2 B + 9\cos^2 A + 24\sin B\cos A = 1$, we get $9(\sin^2 A + \cos^2 A) + 16(\cos^2 B + \sin^2 B) + 24(\sin A\cos B + \sin B\cos A) = 37$, i.e. $25 + 12\sin(A+B) = 37$. Hence, $\sin(A+B) = \frac{1}{2}$, which implies $A+B = 30°$ or $A+B = 150°$. If $A+B = 30°$, then $\angle A < 30°$, so $3\sin A + 4\cos B < \frac{3}{2} + 4 < 6$. Hence, $A+B = 150°$ and $\angle C = 30°$.

(14) Answer: (b). Let x_1 and x_2 be zeros of $x^2 + mx + n = 0$. Then x_1^3 and x_2^3 are the zeros of $x^2 + px + q = 0$. We have: $x_1 + x_2 = -m$, $x_1x_2 = n$, $x_1^3 + x_2^3 = -p$, and $x_1^3x_2^3 = q$. Then $(x_1+x_2)^3 = (x_1^3 + x_2^3) + 3x_1x_2(x_1+x_2)$ implies $-m^3 = -p - 3mn$, i.e. $p = m^3 - 3mn$.

(15) Answer: (e). Let x and y be positive integers. Since $gcd(x,y) = 12$, we have $x = 12m$, $y = 12n$ for some positive integers m and n. From $xy = gcd(x,y)lcd(x,y)$ we get $mn = 66 = 1\cdot 2\cdot 3\cdot 11$. Therefore, $(m,n) \in \{(1,66),(2,33),(3,22),(6,11),(11,6),(22,3),(33,2),(66,1)\}$, i.e. $(x,y) \in \{(12,792),(24,396),(36,264),(72,132),(132,72),(264,36),(396,24),(792,12)\}$.

(16) Answer: (a). $50! = 2^{47}\cdot 3^{22}\cdot 5^{12}\cdot 7^8\cdot 11^4\cdot 13^3\cdot 17^2\cdot 19^2\cdot 23^2\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47 = 10^{12}\cdot m$, where $m = 2^{35}\cdot 3^{22}\cdot 7^8\cdot 11^4\cdot 13^3\cdot 17^2\cdot 19^2\cdot 23^2\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47$. The last digit of m is 2, which is the 13th digit of 50!.

(17) Answer: (c). We immediately see that $x > -1$ and $kx > 0$. From the equation $\log kx = 2\log(x+1)$ we get $x^2 - (k-2)x + 1 = 0$, which has one solution if $k = 4$ or $k = 0$. Since $kx > 0$, we have $k \neq 0$. The quadratic equation has: two real solutions for $k > 4$; no solutions if $k \in (0,4)$; only one real solution of $k = 4$; and two real solutions of $k < 0$. However, in the last case, the quadratic equation has only one root in the interval $(-1,0)$, so for $k < 0$ the given equation has also only one real solution. Therefore, the equation has only one real solution of $k = 4$ or $k < 0$.

(18) Answer: (e). $f(x) + f(1-x) = \frac{4^x}{4^x+2} + \frac{4^{1-x}}{4^{1-x}+2} = \frac{4^x}{4^x+2} + \frac{2}{2+4^x} = 1$. Using this relation, we have $\left(f\left(\frac{1}{2012}\right) + f\left(\frac{2011}{2012}\right)\right) + \cdots + \left(f\left(\frac{1005}{2012}\right) + f\left(\frac{1007}{2012}\right)\right) + f\left(\frac{1006}{2012}\right) = 1005 + f\left(\frac{1}{2}\right) = 1005.5$.

(19) Answer: (c). The remainder has degree at most 1, so $x^{100} - 2x^{51} + 1 = P(x)(x^2-1) + ax + b$. If we substitute $x = 1$, we get $a + b = 0$, and if we substitute $x = -1$ we get $-a + b = 4$. Hence, $a = -2, b = 2$, and the remainder is $-2x + 2$.

(20) Answer: (e). We will determine all values of a such that the graph of the function $y = |x^2+2x+a|$ has four intersections with the line $y = 2$. Since $x^2+2x+a = (x+1)^2 + a - 1$, we need to consider three cases: $a - 1 = 0$, $a - 1 > 0$, and $a - 1 < 0$. If $a - 1 = 0$, then $y = (x+1)^2$ has two intersection points with $y = 2$. If $a - 1 > 0$, then $y = |(x+1)^2 + (a-1)| = (x+1)^2 + (a-1)$ which has 0, 1, 0r 2 intersection points with $y = 2$. If $a - 1 < 0$, the graph of $y = |(x+1)^2 + (a-1)|$ has 2 (if $|a-1| < 2$), 3 ($|a-1| = 2$), or 4 (if $|a-1| > 2$) intersection points with $y = 2$. From $|a-1| > 2$ and $a - 1 < 0$, we get $a < -1$. So the equations has four solutions for $a < -1$.

PART II: 10 INTEGER ANSWER PROBLEMS

(1) Answer: 18. $\frac{1}{n}\left((n-1)+(n-2)+\cdots+2+1\right)=\frac{3n-20}{4}$ is equivalent to $\frac{1}{n}\cdot\frac{(n-1)n}{2}=\frac{3n-20}{4}$. The solution of the last equation is $n=18$.

(2) Answer: 145. $78445=5\cdot 29\cdot 541=145\cdot 541$.

(3) Answer: 675. Let x be the length of the trip. Then $x=100\cdot\left(\frac{180}{80}+\frac{x-180}{110}\right)$. hence $x=675$ km.

(4) Answer: 3. The region is enclosed by the polygon with vertices: $A(1,0)$, $B(0,1)$, $C(-1,1)$, $D(-1,0)$, $E(0,-1)$, and $F(1,-1)$.

(5) Answer: 2. Extend the side $\overline{CA}$ at A to a point E such that the triangle $\triangle ABE$ is equilateral. Then $\frac{AD}{BE}=\frac{CA}{CE}$. Hence $AD=2$.

(6) Answer: 7. If x is a solution of the equation $px^2-qx+p=0$, then $x>0$ (if $x<0$, then $px^2-qx+p>0$). Let $p=\frac{m}{n}$ where m and n are relatively prime positive integer numbers. Then $m(qn-pm)=pn^2$ which implies that $m|p$. Hence, $m=1$ or $m=p$ and $n=1$ or $n=p$. We get $x=1$ or $x=p$ or $x=\frac{1}{p}$. If $x=1$, then $q=2p$ which is not possible since q is prime. If $x=p$ or $x=\frac{1}{p}$, we get $q=p^2+1$. So p must be even. Hence $p=2$ and $q=5$.

(7) Answer: 6. Let x be the number of red balls and y be the number of blue balls. Let $P(A)$ be the probability that both balls are red, $P(B)$ be the probability that the balls are of different color, and $P(C)$ be the probability that both balls are blue. Then
$P(A)=\frac{\binom{x}{2}}{\binom{x+y}{2}}=\frac{x(x-1)}{(x+y)(x+y-1)}$, $P(B)=\frac{\binom{x}{1}\binom{y}{1}}{\binom{x+y}{2}}=\frac{2xy}{(x+y)(x+y-1)}$, $P(C)=\frac{\binom{y}{2}}{\binom{x+y}{2}}=\frac{y(y-1)}{(x+y)(x+y-1)}$.
Since $P(A)=5P(C)$ and $P(B)=6P(C)$, we get $x(x-1)=5y(y-1)$ and $2xy=6y(y-1)$. Hence $x=6$.

(8) Answer: 111. $\log_6(abc)=\log_6 a+\log_6 b+\log_6 c=6$, so $abc=6^6$. Since $ac=b^2$, we get that $b^3=6^6$, i.e. $b=36$. Since $a-b$ is a square of an integer, we get $a\in\{11,20,27,32,35\}$. Since $a|b^2$, we get $a=27$, which implies $c=48$. Therefore, $a+b+c=111$.

(9) Answer: 3. $11^{100}=(10+1)^{100}=\sum_{k=0}^{100}\binom{100}{k}10^{100-k}=\sum_{k=0}^{96}\binom{100}{k}10^{100-k}+\binom{100}{97}10^3+\binom{100}{98}10^2+\binom{100}{99}10+1=10^4\sum_{k=0}^{96}\binom{100}{k}10^{96-k}+33\cdot 4910^4+495\cdot 10^3+1=10^4\sum_{k=0}^{96}\binom{100}{k}10^{96-k}+33\cdot 4910^4+496\cdot+1$. Hence, 11^{100} ends on 6001, which implies that 11^{100} ends with 3 zeros.

(10) Answer: 4. $\frac{3y^2-4xy}{x^2+y^2}=4-\frac{(y+2x)^2}{x^2+y^2}$. This expression will have maximum value if $y+2x=0$, and the maximum value is 4.

TIE BREAKER PROBLEM

Answer: 3. The function will have x-intercepts at points that satisfy $|x-2|-a=3$ or $|x-2|-a=-3$, i.e. $|x-2|=a+3$ or $|x-2|=a-3$. In order for the function to have three x-intercepts one of the last two equation should have exactly one x-intercept and the other equation should have exactly two x-intercepts. Hence, either $a+3>0$ and $a-3=0$ or $a+3=0$ and $a-3>0$. The second alternative is not possible and from the first one we get $a=3$.