The Thirty-fourth Annual State High School Mathematics Contest

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NC STATE MATHEMATICS CONTEST
APRIL 2012

PART I: 20 MULTIPLE CHOICE PROBLEMS

(1) The operation $\circ$ is defined by $x \circ y = 4x - 3y + xy$ for all real numbers x and y. For how many real numbers y does $3 \circ y = 12$?

(a) 1 (b) 2 (c) 4 (d) infinitely many (e) none of (a) through (d) is correct

(2) Find the sum $100^2 - 99^2 + 98^2 - 97^2 + \cdots + 2^2 - 1^2$.

(a) 5050 (b) 5151 (c) 5052 (d) 4950 (e) none of (a) through (d) is correct

(3) Find the sum of all integer numbers n for which $\frac{19n+7}{7n+11}$ is an integer number.

(a) 3 (b) -14 (c) -1 (d) 0 (e) none of a) through d) is correct

(4) Line segments drawn from the vertex opposite the hypothenuse of a right triangle to the points trisecting the hypothenuse have lengths $\sin x$ and $\cos x$, where x is a real number such that $0 < x < \frac{\pi}{2}$. The length of the hypothenuse is

(a) $\frac{4}{3}$ (b) $\frac{3}{2}$ (c) $\frac{3\sqrt{5}}{5}$ (d) $\frac{2\sqrt{5}}{3}$ (e) $\frac{2\sqrt{5}}{5}$

(5) What is the probability that a randomly chosen divisor of 2025 is divisible by 5?
(a) $\frac{3}{5}$ (b) $\frac{11}{15}$ (c) $\frac{2}{3}$ (d) $\frac{8}{15}$ (e) $\frac{7}{15}$

(6) How many ordered pairs of positive integer numbers (a, b) satisfy the equation

$$a + b + ab = 2012?$$

(a) 4 (b) 8 (c) 3 (d) 6 (e) 2

(7) Let $z_0 = 1 - i$ and $z_n = \frac{z_{n-1}+i}{z_{n-1}-i}$ for all positive integer numbers n. Find z_{2012}.

(a) $1 - i$ (b) $-2 - i$ (c) $\frac{1+2i}{5}$ (d) $\frac{1-2i}{5}$ (e) $-2 + i$

(8) Three mutually tangent spheres of radius 1 rest on a horizontal plane. A sphere of radius 2 rests on the top of them. What is the distance from the plane to the top of the larger sphere?

(a) $3 + \frac{\sqrt{30}}{2}$ (b) $3 + \frac{\sqrt{69}}{3}$ (c) $3 + \frac{\sqrt{123}}{4}$ (d) $3 + 2\sqrt{2}$

(e) none of (a) through (d) is correct

(9) Find $1! \cdot 3 - 2! \cdot 4 + 3! \cdot 5 - 4! \cdot 6 + \cdots - 2012! \cdot 2014 + 2013!$.

(a) 0 (b) 2014! (c) 1 (d) 2013!+1 (e) none of (a) through (d) is correct

(10) Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(1) + f(2) + \cdots + f(n) = n^2 f(n)$ for all $n \geq 2$ and $f(1) = 1010$. Find the value of $f(2012)$.

(a) $\frac{505}{1012539}$ (b) $\frac{1010}{2012 \cdot 2013}$ (c) $\frac{505}{1012036}$ (d) $\frac{2020}{503 \cdot 2013}$

(e) none of (a) through (d) is correct

(11) Andrew and Jordan play until one of them wins 5 games. The winner is the one that will win 5 games. The probability that Andrew will win each game is $\frac{2}{3}$ and the probability that Jordan will win each game is $\frac{1}{3}$. What is the probability that the overall winner is determined in exactly 7 games?

(a) $\frac{80}{729}$ (b) $\frac{160}{729}$ (c) $\frac{25}{81}$ (d) $\frac{20}{81}$ (e) none of (a) through (d) is correct

(12) Let $a_1, a_2, a_3, \ldots$ be an arithmetic sequence. Let S_k be the sum of the first k terms of this sequence. If $\frac{S_m}{S_n} = \frac{m^2}{n^2}$, then $\frac{a_m}{a_n}$ is equal to

(a) $\frac{m}{n}$ (b) $\frac{2m}{2n}$ (c) $\frac{2m-1}{2n-1}$ (d) $\frac{m+1}{n+1}$ (e) $\frac{2m+1}{2n+1}$

(13) In $\triangle ABC$ we have $3 \sin A + 4 \cos B = 6$ and $4 \sin B + 3 \cos A = 1$. What is the degree measure of $\angle C$?

(a) $30°$ (b) $45°$ (c) $60°$ (d) $150°$ (e) none of (a) through (d) is correct

(14) The solutions of the equation $x^2 + px + q = 0$ are the cubes of the solutions of the equation $x^2 + mx + n = 0$. Which of the following is true?

(a) $p = m^3 + 3mn$ (b) $p = m^3 - 3mn$ (c) $p = 3mn - m^3$ (d) $p + q = m^3$

(e) none of a) through d) is correct

(15) How many ordered pairs of positive integer numbers have the greatest common divisor 12 and the least common multiple 792?

(a) 4 (b) 5 (c) 6 (d) 3 (e) 8

(16) Find the thirteenth digit from right to left in 50!.

(a) 2 (b) 8 (c) 9 (d) 1 (e) 7

(17) Find all values of k such that the equation $\log(kx) = 2\log(x+1)$ has exactly one real solution.

(a) $k = 4$ (b) $k < 0$ (c) $k = 4$ or $k < 0$ (d) $k = 4$ or $k = -1$

(e) none of a) through d) is correct

(18) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \frac{4^x}{4^x+2}$ for every $x \in \mathbb{R}$. Find the value of

$$f\left(\frac{1}{2012}\right) + f\left(\frac{2}{2012}\right) + f\left(\frac{3}{2012}\right) + \cdots + f\left(\frac{2010}{2012}\right) + f\left(\frac{2011}{2012}\right).$$

(a) 1006.5 (b) 1007 (c) 1007.5 (d) 1006 (e) 1005.5

(19) What is the remainder when $x^{100} - 2x^{51} + 1$ is divided by $x^2 - 1$?

(a) $2x + 2$ (b) $2x - 2$ (c) $-2x + 2$ (d) $-2x - 2$ (e) 2

(20) Determine all values of a such that the equation $|x^2 + 2x + a| = 2$ has exactly four real solutions.

(a) $a > 3$ (b) $a > 2$ or $a \leq -1$ (c) $a > 3$ or $a \leq -3$

(d) $a > 3$ or $a < -1$ (e) $a < -1$

PART II: 10 INTEGER ANSWER PROBLEMS

(1) Find a positive integer number n such that

$$\frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} + \cdots + \frac{2}{n} + \frac{1}{n} = \frac{3n-20}{4}.$$

(2) Let A be a positive integer and B be the integer obtained when A is written in the reverse order. Find the smaller number of A and B if $A \cdot B = 78445$.

(3) Zach drives the first 180 miles of his trip with speed 80 km/h , and the rest of the trip with 110 km/h. The average speed for the whole trip is 100 km/h. Find the length of the trip in km.

(4) Find the area of the region in the xy-plane $\{(x, y) \in \mathbb{R}^2 : |x| + |y| + |x + y| \leq 2\}$.

(5) Point D is on side $\overline{BC}$ of $\triangle ABC$ with $AB = 3$, $AC = 6$, and $m(\angle CAD) = m(\angle DAB) = 60°$. What is the length of AD?

(6) Let p and q be prime numbers such that the equation $px^2 - qx + p = 0$ has rational roots. Find $p + q$.

(7) From a box with red and blue balls we randomly choose two balls. Assume that the box has at least two balls and at least one of them is blue. The probability that the chosen balls are both red is five times the probability that the chosen balls are both blue and the probability that the chosen balls are of different color is six times the probability that the chosen balls are both blue. How many red balls are in the box?

(8) Let a, b, and c be distinct positive real numbers that form a geometric sequence. Let $a - b$ be a square of an integer number and $\log_6 a + \log_6 b + \log_6 c = 6$. Find the value of $a + b + c$.

(9) How many zeros does $11^{100} - 1$ end with?

(10) Let x and y be real numbers. Find the maximum value of the expression $\frac{3y^2 - 4xy}{x^2 + y^2}$.

The following problem, will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test.

TIE BREAKER PROBLEM

For which value of a does the graph of the function $f(x) = ||x - 2| - a| - 3$ has exactly three x-intercepts?