Solutions

PART I: 20 MULTIPLE CHOICE PROBLEMS

(1) Answer: (a). $x^3+y^3=(x+y)^3-3xy(x+y)=125-15=110$.

(2) Answer: (a). From the given equations we have $2^{5x-y}=2^4$ and $5^{y-2x}=5^2$, which implies $5x-y=4$ and $-2x+y=2$. The solution of the system is $x=2$ and $y=6$. Therefore, $xy=12$.

(3) Answer: (c). Let a, aq, aq^2 be the three dimensions of rectangular solid. From $V=8$ we have $a^3q^3=8$, which gives $aq=2$. The surface area is $S=2a^2q+2a^2q^2+2a^2q^3$. Since $S=32$, we have $2aq(a+aq+aq^2)=32$. The last equation together with $aq=2$ gives $a+aq+aq^2=8$. Hence, the sum of the lengths of all edges of the solid is 32 cm.

(4) Answer: (c). We have $x_1+x_2=-\frac{b}{a}$ and $x_1x_2=\frac{c}{a}$. Then $ax_1+b+ax_2+b=a(x_1+x_2)+2b=b$ and $(ax_1+b)(ax_2+b)=a^2x_1x_2+abx_1+abx_2+b^2=ac$. Hence, $x^2-bx+ac=0$ is a quadratic equation with roots ax_1+b and ax_2+b.

(5) Answer: (c). The probability of rolling a five or six is $\frac{1}{3}$. If we roll the die six time, then the probability of rolling a five or six is $\left(\frac{1}{3}\right)^6$. If we roll the die six times, the probability of rolling a five or six exactly five times is $6\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)$. Hence, the probability of rolling a five or six at least five times is $6\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)+\left(\frac{1}{3}\right)^6=\frac{13}{729}$.

(6) Answer: (b). The numbers are divisible by four if their last to digits are: 12, 24, 32, 44, or 52. The first two digits might be any of the given five digits. Therefore, the number of four-digit numbers formed from the digits 1,2,3,4,5 (with possible repetition) that are divisible by 4 is $5\cdot 5^2=125$.

(7) Answer: (a). Let $\alpha=\arctan x$ and $\beta=\arctan y$ where $-\frac{\pi}{2}<\alpha,\beta<\frac{\pi}{2}$. Then $\tan\alpha=x$ and $\tan\beta=y$. Hence $(\tan\alpha+1)(\tan\beta+1)=2$ implies $\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=1$. Since $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$, we have $\tan(\alpha+\beta)=1$. Hence $\alpha+\beta=\frac{\pi}{4}$. Since $x=\frac{\sqrt{3}}{3}$, we have $\alpha=\frac{\pi}{6}$ which together with $\alpha+\beta=\frac{\pi}{4}$ gives $\beta=\frac{\pi}{12}$, i.e. $\arctan y=\frac{\pi}{12}$.

(8) Answer: (c). Since $13!=13\cdot 11\cdot 7\cdot 5^2\cdot 12^5$, it follows that $13!$ ends with 5 zeros in base 12.

(9) Answer: (e). If $x\le -2$, then $f(x)=3-4x$, and $f_{min}=11$ achieved at $x=-2$. If $-2<x\le -1$, then $f(x)=11$, and $f_{min}=11$ achieved at any point from the interval $(-2,-1]$. If $-1<x\le 2$, then $f(x)=-2x+9$, and $f_{min}=5$ achieved for $x=2$. If $x>2$, then $f(x)=4x-3$, and the function does not have a minimum value on this interval. Therefore, the minimum value of f is 5.

(10) Answer: (a). Since the equality holds for every positive integer number n, we have $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n}=\frac{An+B}{2^n}+C$ and $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n}+\frac{n+1}{2^{n+1}}=\frac{A(n+1)+B}{2^{n+1}}+C$. From the last two equations we get $\frac{An+B}{2^n}+C+\frac{n+1}{2^{n+1}}=\frac{A(n+1)+B}{2^{n+1}}+C$, i.e. $(A+1)n+(B-A+1)=0$ for every positive integer n. Hence, $A=-1$ and $B=-2$. If we substitute $n=1$ in the given identity, we have $\frac{A+B}{2}+C=\frac{1}{2}$, which implies $C=2$. Therefore, $A+B+C=-1$.

(11) Answer: (e). Let $z=a+bi$ where a and b are real numbers. Then $z\bar{z}=a^2+b^2$, $z-\bar{z}=2bi$, and from $z\bar{z}+1=-i(z-\bar{z})$ we get $a^2+b^2+1=2b$. The last equation is equivalent to $a^2+(b-1)^2=0$. Hence, $a=0$, $b=1$, and $z=i$.

(12) Answer: (a). The given equation is equivalent to $\log(x-2y)^2=\log(xy)$ which implies $(x-2y)^2=xy$. The last equation is equivalent to $\left(\frac{x}{y}\right)^2-5\left(\frac{x}{y}\right)+4=0$. We get $\frac{x}{y}=4$ or $\frac{x}{y}=1$. Since $x-2y>0$, it follows that $\frac{x}{y}$ cannot be 1. Therefore, $\frac{x}{y}=4$.

(13) Answer: (d). Notice that $f(n)=(n^2-n+1)(n^2+3n+1)$. If $|f(n)|$ is prime, then we need one of the factors to be ± 1. If $n^2-n+1=1$, then $n=0$ or $n=1$; in this case $f(0)=1$, which is not prime, and $f(1)=5$, which is prime. The equation $n^2-n+1=-1$ has no

integer solutions. If $n^2+3n+1=1$, then $n=0$ or $n=-3$; in this case $f(0)=1$, which is not prime, and $f(-3)=13$, which is prime. If $n^2+3n+1=-1$, then $n=-1$ or $n=-2$; in this case $f(-1)=-3$, which is not prime, and $f(-2)=-7$, and both $|f(-1)|$ and $|f(-2)|$ are prime. Hence, $|f(n)|$ is prime for $n=\{-3,-2,-1,1\}$.

(14) Answer: (c). Since $(x-1)(x-5)=x^2-6x+5$ and $(x-2)(x-4)=x^2-6x+8$, the given equation is equivalent to $(x^2-6x+5)(x^2-6x+8)=40$. Let $y=x^2-6x+5$. Then we have $y(y+3)=40$, and the solutions of this equation are $y_1=5$ and $y_2=-8$. From $x^2-6x+5=5$ and $x^2-6x+5=-8$ we have that the solutions of the give equation are $x_1=0$, $x_1=6$, $x_3=3+2i$, and $x_4=3-2i$. The sum of the real solutions of this equation is 6.

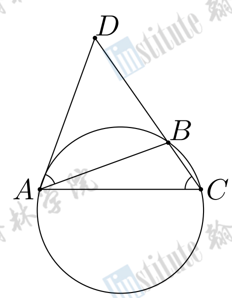
(15) Answer: (b). Let $t=2^x$. Then $\left(t^3-\frac{8}{t^3}\right)-6\left(t-\frac{2}{t}\right)=1$, i.e. $\left(t-\frac{2}{t}\right)^3=1$. Hence $t=-1$ or $t=2$. Since $t=2^x$, it follows that t cannot be negative. Therefore, $t=2$, which implies that $x=1$. Indeed, $x=1$ is the solution of the equation.

(16) Answer: (b). Since $x^2+4x+y^2=0$ is equivalent to $(x+2)^2+y^2=4$, the curve is a circle with center $S(-2,0)$ and radius 2. Let $N(x,y)$ be a point of the circle closest to P. Then $\overline{NP}=\overline{MP}$, $\overline{SP}=\overline{SN}+\overline{NP}=2+\overline{MP}$. Since $\overline{SP}^2=(u+2)^2+v^2$, $\overline{MP}^2=(u-2)^2+v^2$. Hence, $\sqrt{(u+2)^2+v^2}=2+\sqrt{(u-2)^2+v^2}$, which implies $u^2-\frac{v^2}{3}=1$.

(17) Answer: (d). The equation $\frac{\sin^4 x}{a}+\frac{\cos^4 x}{b}=\frac{1}{a+b}$ is equivalent to $\frac{(1-\cos^2 x)^2}{a}+\frac{\cos^4 x}{b}-\frac{1}{a+b}=0$, i.e. $(a+b)\cos^4 x-2b\cos^2 x+\frac{b^2}{a+b}=0$. From the last equation we have $\cos^2 x=\frac{b}{a+b}$. Hence, $\cos^8 x=\frac{b^4}{(a+b)^4}$ and $\sin^8 x=\frac{a^4}{(a+b)^4}$. Therefore, $\frac{\sin^8 x}{a^3}+\frac{\cos^8 x}{b^3}=\frac{a}{(a+b)^4}+\frac{b}{(a+b)^4}=\frac{1}{(a+b)^3}$.

(18) Answer: (c). The equation $x^2+bx+c=0$ has positive real solutions if and only if $\sqrt{b^2-4c}$ is real and $-b-\sqrt{b^2-4c}>0$ which is equivalent to $0\le b^2-4c<b^2$ and $b<0$. The roots are equal if and only if $b^2=4c$. Hence, the equation has positive real distinct roots if and only if $(b,c)\in\{(-3,1),(-3,2),(-4,1),(-4,2),(-4,3),(-5,1),(-5,2),(-5,3),(-5,4),(-5,5)\}$. Therefore, the probability that the equation $x^2+px+q=0$ will not have distinct positive real solutions is $1-\frac{10}{11^2}=\frac{111}{121}$.

(19) Answer: (c)
Since $\angle ACB=\angle BAD$ we have that the triangles ACD and ABD are similar. Hence $\frac{\overline{AC}}{\overline{AB}}=\frac{\overline{AD}}{\overline{BD}}=\frac{\overline{CD}}{\overline{AD}}$. Since $\overline{AB}=1$, we have $\overline{AC}=\frac{\overline{AD}}{\overline{BD}}=\frac{2\overline{BD}}{\overline{AD}}$, i.e. $\left(\frac{\overline{AD}}{\overline{BD}}\right)^2=2$. Therefore $\overline{AC}=\frac{\overline{AD}}{\overline{BD}}=\sqrt{2}$ cm.



(20) Answer: (e). Let $a=x-y$. Substituting $x=a+y$ in $2(x^2+y^2)=x+y$ we have $2(y+a)^2+2y^2=a+2y$, i.e. $4y^2+2(2a-1)y+(2a^2-a)=0$. The last equation has at least one real root if and only if $4(2a-1)^2-16(2a^2-a)\ge 0$, i.e. $4a^2-1\le 0$. The last inequality is satisfied for $a\in\left[-\frac{1}{2},\frac{1}{2}\right]$. Hence, the maximum value of a, and therefore for $x+y$, is $\frac{1}{2}$.

PART II: 10 INTEGER ANSWER PROBLEMS

(1) Answer: 3. The positive integer n must be even. Let $n = 6q + r$ for some integers q and r such that $0 \le r < 6$. Since n is even, r must be even. We also have that $r \neq 0$ and $r \neq 2$, because the remainder when n is divided by 3 is 1. We conclude that $r = 4$, i.e. $n = 6q + 4$. Therefore, $n - 1 = 6q + 3$, which implies that the remainder when $n - 1$ is divided by 6 is 3.

(2) Answer: 4913. Since $y \le 9$, we have $t \le 3$. The cases $t = 0$ and $t = 1$ are not possible, because they imply $y = 0$ and $y = 1$ respectively, which contradicts the hypothesis that all four digits are distinct.

- If $t = 2$, then $y = 4$ and $2x = 4 - z$; the last equation implies that $z < 4$ and z is even. It easy to check that $z = 0$ implies $x = t = 2$; $z = 2$ is also not possible because $t = 2$.
- If $z = 3$, then $y = 9$ and $2x = 9 - z$; the last equation implies that z is odd and $z \neq 9$. If $z = 1$, then $x = 4$; we have $\overline{xyzt} = 4913$ and the sum of the digits is 17. If $z = 3$, then $x = 3$, which is not possible. If $z = 5$, then $x = 2$; we have $\overline{xyzt} = 2953$, but the sum of the digits is not 17. If $z = 7$, then $x = 1$; we have $\overline{xyzt} = 1973$, but the sum of the digits is not 17. Therefore, there is only one four digit number that satisfies all the hypotheses of the question, and that number is 4913.

(3) Answer: 7825. There are $1 + 2 + 3 + \cdots + k - 1 = \frac{k(k-1)}{2}$ elements before the smallest element in the kth set. Hence, the first element in the kth set is $\frac{k(k-1)}{2} + 1$. The elements in the kth set form an arithmetic sequence with first element $\frac{k(k-1)}{2} + 1$ and difference 1. Since there are k elements in the kth set, the sum of the elements in the kth set is $S_k = \frac{k}{2}\left(2\left(\frac{k(k-1)}{2} + 1\right) + (k-1)\right) = \frac{k^3+k}{2}$. Therefore, $S_{25} = 7825$.

(4) Answer: 1350. We can choose exactly two digits on $\frac{10 \cdot 9}{2}$ ways. From the chosen two digits, we can form $2^5 - 2$ sequences that contain both digits (we subtract the two sequences that contain only one of the chosen digits). Hence, the number of sequences of length five with the property that exactly two of the ten digits appear is $\frac{10 \cdot 9}{2}(2^5 - 2) = 1350$.

(5) Answer: 1. $\sqrt{7-\sqrt{48}} + \sqrt{5-\sqrt{24}} + \sqrt{3-\sqrt{8}} = \sqrt{(2-\sqrt{3})^2} + \sqrt{(\sqrt{3}-\sqrt{2})^2} + \sqrt{(\sqrt{2}-1)^2} = 2 - \sqrt{3} + \sqrt{3} - \sqrt{2} + \sqrt{2} - 1 = 1$.

(6) Answer: 6. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be the prime factorization of n such that $p_1, p_2, \ldots, p_k$ are prime numbers and $p_1 < p_2 < \ldots < p_k$. Then the number of divisors of n, denoted by $d(n)$, is $d(n) = (a_1+1)(a_2+1)\cdots(a_k+1)$. In our case $d(n) = 30 = 2 \cdot 3 \cdot 5$, which implies that n has at most three prime factors. Since n is divisible by 30, we have $n = 2^{a_1} 3^{a_2} 5^{a_3}$ where $(a_1+1)(a_2+1)(a_3+1) = 30$. Therefore, the follwoing numbers are divisible by 30 and have exactly 30 divisors: $2 \cdot 3^2 \cdot 5^4$, $2 \cdot 3^4 \cdot 5^2$, $2^2 \cdot 3 \cdot 5^4$, $2^2 \cdot 3^4 \cdot 5$, $2^4 \cdot 3 \cdot 5^2$, and $2^4 \cdot 3^2 \cdot 5$.

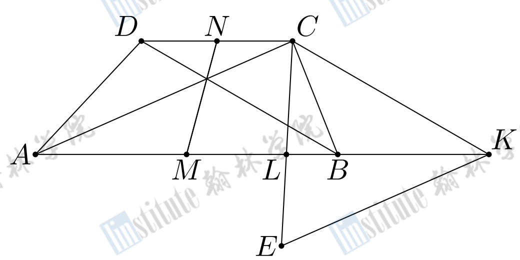
(7) Answer: 8. The given equation is equivalent to $(a+b)^2 + b^2 = 13$. Since a and b are integers, we have that $-3 \le a + b \le 3$. If $a + b = \pm 1$ or $a + b = 0$ we have $b^2 = 12$ and $b^2 = 13$ respectively, and the last two equation do not have integer solutions. We need to consider the following four cases: $a + b = \pm 3$ and $a + b = \pm 2$. If $a + b = -3$, we have $b^2 = 4$; the solutions in this case are $(-5, 2)$ and $(-1, -2)$. If $a + b = 3$, we have $b^2 = 4$; the solutions in this case are $(1, 2)$ and $(5, -2)$. If $a + b = -2$, we have $b^2 = 9$; the solutions in this case are $(-5, 3)$ and $(1, -3)$. If $a + b = 2$, we have $b^2 = 9$; the solutions in this case are $(-1, 3)$ and $(5, -3)$.

(8) Answer: 6. Let x be the number of matches in which women defeated men. The number of matches between two women is $\frac{n(n-1)}{2}$, the number of matches between two men is $\frac{2n(2n-1)}{2}$, and the number of matches between a man and a woman is $2n^2$. The number of matches won by women is $\frac{n(n-1)}{2} + x$, and the number of matches won by men is $\frac{2n(2n-1)}{2} + 2n^2 - x$. Hence, $\left(\frac{n(n-1)}{2} + x\right) : \left(\frac{2n(2n-1)}{2} + 2n^2 - x\right) = 7 : 5$. The last equation is equivalent to

$8x = 17n^2 - 3n$. Since $x \le 2n^2$, we have $8x \le 16n^2$. This implies $17n^2 - 3n \le 16n^2$, i.e. $n(n-3) \le 0$. Hence, the possible values for n are 1,2, or 3. For $n = 1$ and $n = 2$ the equation $8x = 17n^2 - 3n$ does not have integer solution, and for $n = 3$ we have $x = 18$. Therefore, six men participated in the tournament.

(9) Answer: 225. The sum of the first k terms of the arithmetic sequence is $S_k = \frac{k}{2}(2a_1 + (k-1)\cdot 2) = k(a_1 + k - 1)$. Since $\frac{S_{3n}}{S_n}$ does not depend on n, we have $\frac{S_6}{S_2} = \frac{S_3}{S_1}$. This implies $\frac{3(a_1+2)}{a_1} = \frac{6(a_1+5)}{2(a_1+1)}$, i.e. $a_1 = 1$. Indeed, if the first term of an arithmetic sequence is 1 and its difference is 2, we have $\frac{S_{3n}}{S_n} = \frac{3n(1+3n-1)}{n(1+n-1)} = 9$, which does not depend on n. Therefore, there is only one arithmetic sequence that satisfies the given properties, and that sequence is 1,3,5,... . The sum of the first 15 terms is 225.

(10) Answer: 6. Let $CK || BD$. Then the triangle AKC has the same area as the trapezoid $ABCD$. It is easy to show that $\overline{CL} = \overline{MN} = 2$, and L is the midpoint of the segment AK. Let E be a point on the line CL such that $\overline{CL} = \overline{LE}$ (see the figure). Then the triangles ALC and KLE are congruent. Hence the triangles AKC and CEK have the same area, which is the same as the area of the trapezoid $ABCD$. In $\triangle CEK$ we have $\overline{CE} = 4$, $\overline{EK} = 5$, $\overline{CK} = 3$. Using Heron's formula we have $A_{ABCD} = A_{CEK} = \sqrt{6(6-5)(6-3)(6-4)} = 6$.



TIE BREAKER PROBLEM

Answer: 31. $\sin\alpha = \frac{1-\sqrt{2}}{\sqrt{6-3\sqrt{2}}-\sqrt{2+\sqrt{2}}} = \frac{1-\sqrt{2}}{\sqrt{6-3\sqrt{2}}-\sqrt{2+\sqrt{2}}} \cdot \frac{\sqrt{6-3\sqrt{2}}+\sqrt{2+\sqrt{2}}}{\sqrt{6-3\sqrt{2}}+\sqrt{2+\sqrt{2}}} = \frac{1-\sqrt{2}}{6-3\sqrt{2}-2-\sqrt{2}} \cdot (\sqrt{3}\sqrt{2-\sqrt{2}} + \sqrt{2+\sqrt{2}}) = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} = \cos\frac{\pi}{6} \cdot \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} + \sin\frac{\pi}{6} \cdot \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \cos\frac{\pi}{6} \cdot \sqrt{\frac{1-\cos\frac{\pi}{4}}{2}} + \sin\frac{\pi}{6} \cdot \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} = \cos\frac{\pi}{6}\sin\frac{\pi}{8} + \sin\frac{\pi}{6}\cos\frac{\pi}{8} = \sin\frac{7\pi}{24}$.