NC STATE MATHEMATICS CONTEST
APRIL 2011

PART I: 20 MULTIPLE CHOICE PROBLEMS

(1) Let x and y be real numbers such that $x+y=5$ and $xy=1$. Find x^3+y^3.

(a) 110 (b) 125 (c) 120 (d) 115 (e) 123

(2) If $\frac{4^{3x}}{2^{x+y}}=16$ and $\frac{5^{3y}}{25^{x+y}}=25$, find the value of xy.

(a) 12 (b) 10 (c) -12 (d) 8 (e) 6

(3) The volume of a rectangular solid is 8 cm^3, its surface area is 32 cm^2, and its three dimensions form a geometric sequence. Find the sum of the lengths of all edges of the solid.

(a) 36 cm (b) 30 cm (c) 32 cm (d) 28 cm (e) 34 cm

(4) Let x_1 and x_2 be the roots of $ax^2+bx+c=0$. Find a quadratic equation with roots ax_1+b and ax_2+b.

(a) $x^2+bx+ac=0$ (b) $x^2-bx-ac=0$ (c) $x^2-bx+ac=0$

(d) $x^2+bx-ac=0$ (e) $x^2-ax+bc=0$

(5) A fair die is rolled six times. What is the probability of rolling five or six at least five times?

(a) $\frac{12}{729}$ (b) $\frac{12}{729^2}$ (c) $\frac{13}{729}$ (d) $\frac{1}{729}$ (e) none of (a) through (d) is correct

(6) How many four-digit numbers are there formed from the digits 1,2,3,4,5 (with possible repetition) that are divisible by 4?

(a) 30 (b) 125 (c) 156 (d) 256 (e) none of a) through d) is correct

(7) Let $x=\frac{\sqrt{3}}{3}$ and $(x+1)(y+1)=2$. Find $\arctan y$.

(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$ (e) none of (a) through (d) is correct

(8) With how many zeros does the number 13! (written in base 10) end in base 12?

(a) 3 (b) 10 (c) 5 (d) 4 (e) 2

(9) Let $f(x) = |3x-6| - |x+1| + |2x+4|$ be a function defined on the set of the real numbers. Find the minimum value of f.

(a) 11 (b) f does not have a minimum value (c) -5 (d) 7 (e) 5

(10) Let A, B, and C be real numbers such that $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = \frac{An+B}{2^n} + C$ holds for every positive integer number n. Find $A+B+C$.

(a) -1 (b) 0 (c) -2 (d) 1 (e) 2

(11) Find the product of all complex numbers z that satisfy the equation $z\bar{z}+1 = -i(z-\bar{z})$ $(i = \sqrt{-1})$.

(a) $-1-i$ (b) $1-i$ (c) 1 (d) $-i$ (e) i

(12) Let x and y be positive real numbers such that $x - 2y > 0$.
If $2\log(x-2y) = \log x + \log y$, find $\frac{x}{y}$.

(a) 4 (b) 5 (c) $10^{4\log y}$ (d) $(x+2y)^2$ (e) 3

(13) Let $f(n) = n^4 + 2n^3 - n^2 + 2n + 1$ be a function defined on the set of the integer numbers. For how many integer numbers n is $|f(n)|$ a prime number?

(a) 6 (b) 1 (c) 2 (d) 4 (e) infinitely many

(14) Find the sum of the real solutions of the equation $(x-1)(x-2)(x-4)(x-5) = 40$.

(a) 12 (b) -12 (c) 6 (d) -6 (e) none of (a) through (d) is correct

(15) How many real solutions does the following equation have

$$2^{3x} - \frac{8}{2^{3x}} - 6\left(2^x - \frac{1}{2^{x-1}}\right) = 1?$$

a) 0 b) 1 c) 2 d) 3 e) infinitely many

(16) If the distance between the point $P(u,v)$ and the curve $x^2 + 4x + y^2 = 0$ is the same as the distance between the points $P(u,v)$ and $M(2,0)$, then u and v satisfy the following equation:

(a) $u^2 + \frac{v^2}{3} = 1$ (b) $u^2 - \frac{v^2}{3} = 1$ (c) $\frac{u^2}{3} - v^2 = 1$

(d) $\frac{u^2}{3} + v^2 = 1$ (e) none of (a) through (d) is correct

(17) If $\frac{\sin^4 x}{a} + \frac{\cos^4 x}{b} = \frac{1}{a+b}$, then $\frac{\sin^8 x}{a^3} + \frac{\cos^8 x}{b^3}$ is equal to:

(a) $\frac{1}{a^2+b^2}$ (b) $\frac{1}{(a+b)^2}$ (c) $\frac{1}{a^3+b^3}$ (d) $\frac{1}{(a+b)^3}$ (e) none of (a) through (d) is correct

(18) An ordered pair (b, c) of integers such that $|b| \leq 5$ and $|c| \leq 5$ is chosen at random, with each such ordered pair having an equal likelihood of being chosen. What is the probability that the equation $x^2 + bx + c = 0$ will not have distinct positive real solutions?

(a) $\frac{109}{121}$ (b) $\frac{110}{121}$ (c) $\frac{111}{121}$ (d) $\frac{112}{121}$ (e) none of (a) through (d) is correct

(19) Triangle ABC is inscribed in a circle and $\overline{AB} = 1$ cm. The tangent to the circle at A meets the secant line through B and C at a point D. If B is the midpoint of the segment CD, find the length of the segment AC.

(a) 1 cm (b) $\frac{\sqrt{3}}{2}$ cm (c) $\sqrt{2}$ cm (d) $\sqrt{6}$ cm (e) none of (a) through (d) is correct

(20) Let x and y be real numbers. Find the maximum of $x - y$ if $2(x^2 + y^2) = x + y$.

(a) 1 (b) $\frac{1}{4}$ (c) $\frac{1}{3}$ (d) 2 (e) $\frac{1}{2}$

PART II: 10 INTEGER ANSWER PROBLEMS

(1) When a positive integer n is divided by 3, the remainder is 1. When $n+1$ is divided by 2, the remainder is 1. What is the remainder when $n - 1$ is divided by 6?

(2) Find all four digit positive integer numbers $\overline{xyzt}$ such that all digits are different, $x + y + z + t = 17$, $2x = y - z$, and $y = t^2$.

(3) The set of positive integer numbers is partitioned in the following way:

$$\{1\}, \{2, 3\}, \{4, 5, 6\}, \{7, 8, 9, 10\}, \ldots.$$

Let S_k be the sum of the elements in the kth set. Find S_{25}.

(4) How many sequences of length five can be formed using the digits $0, 1, 2, \ldots, 9$ with the property that exactly two of the ten digits appear? (Leading zeros are allowed and 01110 is an example of a legitimate sequence.)

(5) Find the value of the expression $\sqrt{7 - \sqrt{48}} + \sqrt{5 - \sqrt{24}} + \sqrt{3 - \sqrt{8}}$.

(6) How many positive integer numbers are divisible by 30 and have exactly 30 divisors?

(7) How many ordered pairs of integer numbers (a, b) satisfy the equation

$$a^2 + 2ab + 2b^2 = 13?$$

(8) In a chess tournament, n women and $2n$ men participated, and each one of them played only one game with everybody else. The ratio of the number of games won by women to the number of the games won by men is $7 : 5$. Find the number of men participated in the tournament if no game was over in a tie.

(9) Let $a_1, a_2, a_3, \ldots$ be an arithmetic sequence with difference 2. Let S_k be the sum of the first k terms of this sequence. If $\frac{S_{3n}}{S_n}$ does not depend on n, find the sum of the first 15 terms of this sequence.

(10) Let $ABCD$ be a trapezoid such that $AB||CD$. Let M and N be the midpoints of the sides AB and CD respectively. If the lengths of the diagonals AC and BD of the trapezoid are 5 and 3 respectively, and $\overline{MN} = 2$, find the area of the trapezoid.

The following problem, will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test.

TIE BREAKER PROBLEM

Let $\alpha = \frac{m\pi}{n}$ where m and n are positive relatively prime integer numbers. Find $m + n$ if $0 \leq \alpha \leq \frac{\pi}{2}$ and $\sin\alpha = \frac{1-\sqrt{2}}{\sqrt{6-3\sqrt{2}}-\sqrt{2+\sqrt{2}}}$.