ALGEBRA II
State Mathematics Contest Finals
April 29, 2010

1. If $f(x+1) = f(x-1)+2$ and $f(5)=6$, then $f(1)$ equals

   a. 1 b. 2 c. 4 d. 5 e. 6

2. If the area bounded by the *x*-axis and lines $y = mx + 4$, $x = 1$, and $x = 4$ is 7, then *m* equals:

   a. $-\frac{1}{2}$ b. $-\frac{2}{3}$ c. $-\frac{3}{2}$ d. -2 e. none of these

3. A circle is unwound and re-shaped into a square. Find the ratio of the square's area to the circle's area.

   a. $\frac{\pi}{8}$ b. $\frac{\pi}{4}$ c. $\frac{1}{2}$ d. $\frac{1}{\sqrt{2}}$ e. $\frac{4}{\pi}$

4. There is a list of seven numbers. The average of the first four numbers is 5, and the average of the last four numbers is 8. If the average of all seven numbers is $6\frac{4}{7}$, then the number common to both sets of four numbers is

   a. $5\frac{3}{7}$ b. 6 c. $6\frac{4}{7}$ d. 7 e. $7\frac{3}{7}$

5. If *x* and *y* are positive numbers and the average of 4, 20 and *x* is equal to the average of *y* and 16, then the ratio *x*:*y* is

   a. 3:2 b. 2:3 c. 1:1 d. 2:5 e. 5:2

6. Given $f(x) = x+2$ and $g(x) = \sqrt[3]{x}$, find $\left(f^{-1} \circ g^{-1}\right)(2)$.

   a. 8 b. -6 c. 2 d. -2 e. 6

7. If $\log_b 2 = A$ and $\log_b 5 = c$ where $b > 0$ with $b \neq 1$, then $\log_b 500$ is equal to which of the following?

   a. $3A+2c$ b. $10Ac$ c. A^2+c^3 d. $2A+3c$ e. none of these

8. For how many integers n between 1 and 1990 is the improper fraction $\frac{(n^2+7)}{(n^2+4)}$ NOT in lowest terms?

a. 0 b. 86 c. 90 d. 104 e. 105

9. The first two of three consecutive multiples of 9 sum to 2511. The smallest of these three numbers is

a. 837 b. 936 c. 1134 d. 1251 e. 1305

10. Big Ben's Pizza Shop likes to cover its pizzas with $0.5\,in^3$ of cheese per square *in*. How many whole pizzas with radius 4 *in* can a rectangular block of cheese that measures 4 *in* × 4 *in* × 6 *in* cover?

a. 1 b. 2 c. 3 d. 4 e. more than 4

11. Let $\otimes$ be an operation defined on functions such that $(f \otimes g)(x) = f(g(x)) - g(f(x))$. If $f(x) = x^2 - 1$ and $g(x) = 2x + 1$, then $(f \otimes g)(x)$ equals

a. $x^2 - 2x - 2$ b. $2x^2 - 4x + 1$ c. $2x^2 + 4x - 2$ d. $2x^2 + 4x + 1$ e. $2x^2 + 2x + 1$

12. One solution of $x^3 + 5x^2 - 2x - 4 = 0$ is $x = 1$. Which of the following is another solution?

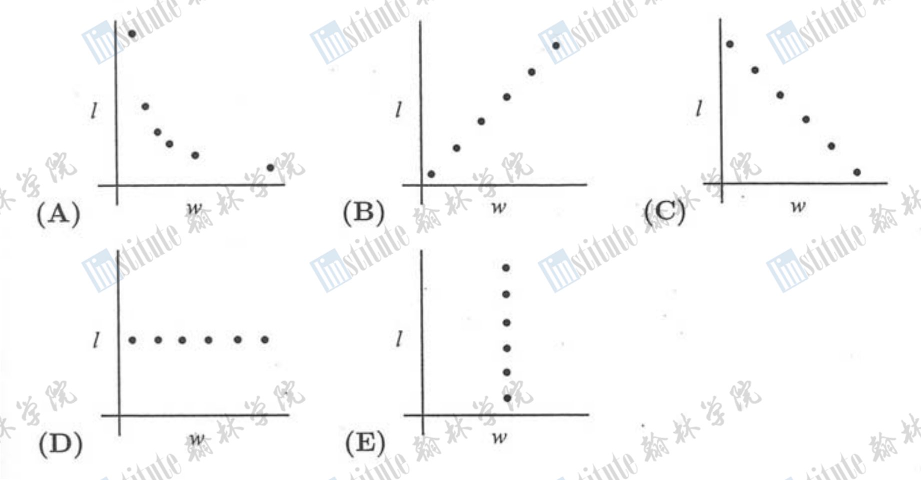
a. $-1+\sqrt{7}$ b. $-3+\sqrt{5}$ c. $-2+\sqrt{5}$ d. $-3+\sqrt{3}$ e. $-5+\sqrt{2}$

13. In the Fibonacci sequence, where $a_1 = 1$, $a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \ge 3$ which are true?

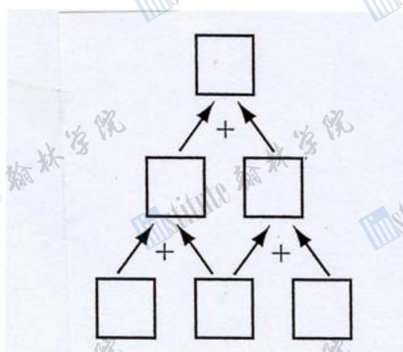
I. a_{2009} is even.
II. a_{2009} is odd.
III. a_{2010} is even.
IV. a_{2010} is odd.

a. I & III b. I & IV c. II & III d. II & IV e. cannot be determined.

14. Bill's teacher asks him to plot all the ordered pairs (w, l) of positive integers for which w is the width and l is the length of a rectangle with area 12. What should his graph look like?



15. Three different one-digit positive integers are placed in the bottom row of cells. Numbers in adjacent cells are added and the sum is placed in the cell above them. In the second row, continue the same process to obtain a number in the top cell. What is the difference between the largest and smallest numbers possible in the top cell?



a. 16 b. 24 c. 25 d. 26 e. 35

16. The squash season is nearing its end, and the current individual standings are shown in the chart. Each of the 8 players must still play 28 games, 4 with each of the other players. How many players still have a theoretical chance to at least tie for the championship?

Player:	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**
Games won:	92	91	90	71	67	66	44	39
Games Lost:	48	49	50	69	73	74	96	101

a. 3 b. 4 c. 5 d. 6 e. 7

17. In a pack of construction paper, the numbers of blue and red sheets are originally in the ratio 2:7. Each day Laura uses 1 blue sheet and 3 red sheets. One day, she uses 3 red sheets and the last blue sheet, leaving her with 15 red sheets. How many sheets of construction paper were in the pack originally?

a. 105 b. 135 c. 171 d. 207 e. 255

18. If p is a multiple of 3 and q is a multiple of 5, which of the following is (are) true?

I. $p+q$ is even
II. pq is odd.
III. $5p+3q$ is a multiple of 15

a. None b. II only c. III only d. I, II only e. II, III only

19. If $\left(a+\frac{1}{a}\right)^2=3$, then $a^3+\frac{1}{a^3}$ equals

a. 0 b. $3\sqrt{3}$ c. $\frac{10\sqrt{3}}{3}$ d. $6\sqrt{3}$ e. $7\sqrt{3}$

20. If the sum of two numbers is 1 and their product is 1, then what is the sum of their cubes?

a. -2 b. $-1+\frac{i\sqrt{3}}{2}$ c. 0 d. $-1-\frac{i\sqrt{3}}{2}$ e. 2

21. When the base of a ladder is 16 *ft* from the base of a wall, 3 *ft* of the ladder projects beyond the top of the wall. When the base of the ladder is 9 *ft* from the base of the wall, 8 *ft* of the ladder projects beyond the top of the wall. How long is the ladder?

a. 15 *ft* b 16.5 *ft* c. 20 *ft* d. 23 *ft* e. 25.25 *ft*

22. If a, b, and c are real numbers such that $\frac{a}{b}=3$ and $\frac{b}{c}=7$, then $\frac{a+b}{b+c}$ equals

a. $\frac{7}{2}$ b. $\frac{7}{8}$ c. $\frac{3}{7}$ d. $\frac{1}{7}$ e. 21

23. An ordered pair (b, c) of integers, each of which has absolute value less than or equal to five, is chosen at random, with each such ordered pair having an equal likelihood of being chosen. What is the probability that the equation $x^2 + bx + c = 0$ will *not* have distinct positive real roots?

a. $\frac{106}{121}$ b. $\frac{108}{121}$ c. $\frac{110}{121}$ d. $\frac{112}{121}$ e. none of these

24. Some students in a gym class are wearing blue jerseys, and the rest are wearing red jerseys. There are exactly 25 ways to pick a team of 3 players that includes at least one player wearing each color. Compute the number of students in the class.

a. 6 b. 7 c. 9 d. 11 e. 12

25. Let $f(x) = ax + b$ were a and b are real numbers, $f(f(f(1))) = 29$ and $f(f(f(0))) = 2$. Then b equals

a. $\frac{2}{7}$ b. 2 c. 3 d. $\frac{2}{13}$ e. 7

26. Find $x + y$ where (x, y) represents the only ordered pair solution of real numbers of the system of equations $\begin{cases} 3^x \cdot 9^y = 81 \\ \frac{2^x}{8^y} = \frac{1}{128} \end{cases}$

a. $\frac{7}{5}$ b. $\frac{9}{5}$ c. $\frac{11}{5}$ d. $\frac{13}{5}$ e. 3

27. Consider all possible four-digit numbers created by using the digits 1, 2, 3, and 4 and using each digit exactly once. What is the sum of these four-digit numbers?

a. 60,000 b. 60,600 c. 60,660 d. 66,000 e. 66,660

28. In a school that has 20 teachers, 10 teach humanities, 8 teach social studies, and 6 teach science. Two teach both humanities and social studies, but none teach both social studies and science. How many teach both humanities and science?

a. 1 b. 2 c. 3 d. 4 e. 5

29. Find the sum of the values of m that make $f(x) = x^2 + (m+5)x + (5m+1)$ a perfect square trinomial.

a. 3 b. 4 c. 7 d. 8 e. 10

30. Consider the equation $x^2 + kx + 1 = 0$. A single fair die is rolled to determine the value of the middle coefficient, k. The value for k is the number of dots on the upper face of the die. The probability that the equation will have real, unequal roots is:

a. $\frac{1}{3}$ b. $\frac{2}{3}$ c. $\frac{1}{2}$ d. $\frac{3}{4}$ e. none of these

31. If the measure of one angle of a rhombus is 60°, then the ratio of the length of its longer diagonal to the length of its shorter diagonal is

a. 2:1 b. $\sqrt{3}:1$ c. $\sqrt{2}:1$ d. $\sqrt{3}:2$ e. $\sqrt{2}:2$

32. The number of common points shared by the graphs of $|x| + |y| = 2$ and $x^2 - y = 2$ are

a. 0 b. 2 c. 3 d. 4 e. 5

33. If $i^2 = -1$, then the sum $i^0 + i^1 + i^2 + i^3 + ... + i^{2009} + i^{2010}$, is

a. 0 b. 1 c. -1 d. i e. $-i$

34. The sum of four numbers labeled A, B, C, and D is 54. Adding 2 to number A, subtracting 2 from number B, doubling number C, and halving number D all result in the same value. What are the four original numbers in increasing numerical order?

a. 1, 2, 4, 47 b. 2, 4, 8, 10 c. 3, 7, 12, 32 d. 6, 10, 14, 24 e. 8, 10, 16, 20

35. If x, y, z satisfy $|x+5| + |y-3| + |z-4| = 1$, which of the following could be $|x + y + z|$?

a. 5 b. -1 c. 16 d. 8 e. 2

36. What is the coefficient of the linear term of the least degree polynomial whose graph passes through (1,1), (0,-1), (2,4)?

a. 0 b. 2 c. $\frac{7}{2}$ d. $\frac{3}{2}$ e. $\frac{11}{2}$

37. Suppose that a and b are positive integers whose greatest common divisor is 2 and whose least common multiple is 40. If $|b-a|=2$, what is the value of $a+b$?

a. 14 b. 16 c. 18 d. 20 e. 22

38. Suppose x, y, z is a geometric sequence with common ratio r, and x is not equal to y. If x, $2y$, $3z$ is an arithmetic sequence, then r is

a. $\frac{1}{4}$ b. $\frac{1}{3}$ c. $\frac{1}{2}$ d. 2 e. 4

39. A cryptographer devises the following method for encoding positive integers. First, the integer is expressed in base 5. Second, a 1-1 correspondence is established between the digits that appear in the expressions in base 5 and the elements in the set V, W, X, Y, Z. Using this correspondence, the cryptographer finds that three consecutive integers in increasing order are coded VYZ, VYX, VVW, respectively. What is the base-10 expression for the integer coded as XYZ?

a. 48 b. 71 c. 82 d. 108 e. 113

40. The following is a Circular Sudoku puzzle. Each of the numbers 1-8 must appear once in every ring and once in every pair of touching slices. Determine the number that lies in the shaded region of the puzzle's solution.

a. 7 b. 6 c. 5 d. 4 e. 3