NC STATE MATHEMATICS CONTEST – APRIL 2009: SOLUTIONS

1. Answer **e**): $A = 2\pi rh = 2\pi r\left(24/\left(\pi r^2\right)\right) = 48/r$ which has no minimum for $r > 0$.

2. Answer **a**): The vertices of the triangle are $(2a/3, 2a/3)$, $(a, 0)$ and $(2a, 2a)$. The area is $2a^2/3$.

3. Answer **a**): If $|AB| = 1$, $ABCDEF$ has area $3\sqrt{3}/2$ and $|AC| = \sqrt{3}$ so ACE has area $3\sqrt{3}/4$.

4. Answer **b**): $\tan^{-1}\left(2\sin\left(2\pi/3\right)\right) = \tan^{-1}\left(\sqrt{3}\right) = \pi/3$.

5. Answer **d**): {TT, HTT, THT, HHTT, HTHT, THHT} has probability 1/4 + 1/4 + 3/16 = 11/16.

6. Answer **b**): $3s/\left(s/10 + s/15 + s/20\right) = 13 + 11/13$.

7. Answer **c**): $x^2 + y^2 = y^2\left(1 + \tan^2(t)\right) = y^2\sec^2(t) = xy\dfrac{y}{x}\sec^2(t) = \dfrac{\sin(2t)}{\tan(t)}\sec^2(t) = \dfrac{2\sin(t)\cos(t)}{\sin(t)\cos(t)} = 2$.

8. Answer **d**): The plane enters the cylinder at (3, 0, 5/2) and leaves the cylinder at (–3, 0, 11/2). The total surface area is that of a cylinder with $r = 3$ and $h = 5/2$ plus one-half that of a cylinder with $r = 3$ and $h = 3$.

9. Answer **d**): With $G =$ the value of the gft: $c + \dfrac{a+b}{2} = b + \dfrac{a+c}{3} = a + \dfrac{b+c}{4} = G$. It follows that the value of c is $5G/17$ so 17 divides G and $17 \times 5.88 = 99.96$.

10. Answer **c**): $\left(x^2 - 1\right)^2 = y^2$ so $y = \pm\left(x^2 - 1\right)$.

11. Answer **b**): $\ln\left(4 + \sqrt{15}\right) = -\ln\left(4 - \sqrt{15}\right)$ and $\ln\left(\dfrac{4 + \sqrt{15}}{4 - \sqrt{15}}\right) = \ln\left(31 + 8\sqrt{15}\right)$.

12. Answer **d**): The mean of the remaining numbers is $\dfrac{N \cdot N - M \cdot M}{N - M} = N + M$.

13. Answer **e**): $2r_1 = 2r_2 + 10$ & $r_2{}^2 = \left(r_1 - r_2\right)^2 + \left(r_1 - 6\right)^2 \Rightarrow r_1 = 18$ & $r_2 = 13$. $18^2 - 13^2 = 155$.

14. Answer **a**): Using $a + b = -3$ & $ab = 5$, $m = \dfrac{(a+2)^2 + (b+2)^2}{(a+2)(b+2)} = \dfrac{-5}{3}$

15. Answer **c**): Only 2 of the $2^8 = 256$ possible colorings have the desired property. 2/256 = 1/128.

16. Answer **d**): The value of the determinant is $6 + y - 5x$, and, with $y = 4 + 5k$, $x = 2 + k$, the determinant is 0.

17. Answer **e**): $-g(x)$ and $g(-x)$ both satisfy (J). $F(-F(x)) = F(g(-x)) = -g(-g(-x))$, if $F(x) = -g(-x)$. Now, $-g(-g(-x)) = -x$ if $g(-x) = -g(x)$ but $-g(-g(-x)) \neq -x$, if, for example, $g(x) = 1 - x$ which does satisfy (I).

OVER FOR SOLUTIONS TO THE REMAINING MULTIPLE CHOICE PROBLEMS AND THE INTEGER ANSWER PROBLEMS.

18. Answer **a)**: It follows from $3r^2 = (a-4)/a$ & $-r^3 = 2/a$ that $r(a-4)/(3a) = r^3 = -2/a$ so that $r = 6/(4-a)$. Then $3(36a) = (a-4)^3$ which has solutions $a = -2$ and 16.

19. Answer **a)**: $1 - \dfrac{(n+2)(n+1)}{(2n+2)(2n+1)} = \dfrac{3n}{4n+2} < \dfrac{3}{4}$ for every n.

20. Answer **b)**: If E is the triangle's vertex on BC, then $|AE| = 1/\cos(\pi/12) = 1/\cos(\pi/3 - \pi/4) = 4/\left(\sqrt{2}+\sqrt{6}\right) = \sqrt{6}-\sqrt{2}$. OR $|BE|^2 + 1 = 2|EC|^2$ & $|BE| + |EC| = 1$. Then $|BE| = 2-\sqrt{3}$ and $|AE| = \sqrt{1+\left(2-\sqrt{3}\right)^2}$ which does equal $\sqrt{6}-\sqrt{2}$.

<u>Integer Answer Problems</u>

1. Answer: **14** Since $i^2 + j^2 = m^2 + n^2$ with $1 \le i \le 4$ & $0 \le j \le i$ and $1 \le m \le 4$ & $0 \le n \le m$, implies that $i = m$ & $j = n$, the number of distinct distances is 2 + 3 + 4 + 5 = 14.
2. Answer: **120** $r = (1 - 1/2 - 1/3 - 1/15)t = t/10$ & $r + 6 = 3t/20 \Rightarrow t = 120$.
3. Answer: **360** $(5!) + C(5,2)(4\cdot 3\cdot 2) = 360$.
4. Answer: **16,000** With a = value of the car, $7(a+c+18000)/12 = a + 5000$ and $8(a+c+18000)/12 = a+c+6000$, so that a = 16,000 and c = 2000.
5. Answer: **20** We want n with number of divisors equal 3, 5, 7, 11, … . [Not 2 because if n has 2 divisors, n is prime.] n must be a prime power if n has a prime number of divisors. Then n must be $2^2, 2^4, 2^6, 2^{10}, 3^2, 3^4, 3^6, 5^2, 5^4$ or p^2 with p = 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43.
6. Answer: **8** $1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 7-a \\ 9-d & d \end{bmatrix} \Rightarrow 9a + 7d = 64$. It follows that $a = d = 4$.
7. Answer: **1089** $1000d_0 + 100d_1 + 10d_2 + d_3 = 9(d_0 + 10d_1 + 100d_2 + 1000d_3) \Rightarrow d_0 = 9$ & $d_3 = 1$**.** Then $100d_1 + 10d_2 + 1 = 81 + 90d_1 + 900d_2 \Rightarrow d_2 = 0$ & $d_1 = 8$.
8. Answer: **300** Let w (h) be the width (height) of the rectangle, and let H be the height of the triangle, with its hypotenuse as base. Then $w/50 = (H-h)/H$. So, $wh = 50(H-h)h/H$ which is maximized when $h = \dfrac{H}{2} = \dfrac{1}{2}\left(\dfrac{(30\cdot 40)/2}{50/2}\right)$.
9. Answer: **1** $\dfrac{z_j}{1+z_j^{\,2}} = \dfrac{z_j^{\,4}}{1+z_j^{\,3}}$ & $\dfrac{z_j^{\,2}}{1+z_j^{\,4}} = \dfrac{z_j^{\,3}}{1+z_j}$ so $w_j = 2\left(\dfrac{z_j}{1+z_j^{\,2}} + \dfrac{z_j^{\,2}}{1+z_j^{\,4}}\right) = 2\left(\dfrac{z_j\left(1+z_j^{\,4}\right) + z_j^{\,2}\left(1+z_j^{\,2}\right)}{\left(1+z_j^{\,2}\right)\left(1+z_j^{\,4}\right)}\right) = 2\left(\dfrac{1+z_j+z_j^{\,2}+z_j^{\,4}}{1+z_j+z_j^{\,2}+z_j^{\,4}}\right) = 2$ for each j.
10. Answer: **8039** There are 2000 integers between 1 and 8000 inclusive whose digit sum is divisible by 4. The next 9 such integers are 8004, 8008, 8013, 8017, 8022, 8026, 8031, 8035 and 8039.

TIE BREAKER ANSWER: 119