NC STATE MATHEMATICS CONTEST – APRIL 2009

PART I: 20 MULTIPLE CHOICE PROBLEMS

1. If C is a right circular cylinder whose volume is 24 cubic inches and r is the radius (in inches) of the cylinder, which of the following statements about A, the area of the curved surface of C, is correct?

a) A is smallest when $r=\sqrt{2}$.

b) A is smallest when $r=2$.

c) A is smallest when $r=3$.

d) A is smallest when $r=\sqrt{6}$.

e) There is no smallest value of A.

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2. If a is a positive real number, what is the area of the region in the first quadrant that is bounded above by the graph of $y = x$ and below by the graph of $y=2|x-a|$?

a) $\frac{2a^2}{3}$ b) a^2 c) $\frac{4a^2}{3}$ d) $2a^2$ e) None of a) through d) is correct.

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3. If A through F are the vertices of a regular hexagon listed in clockwise order, consider the triangle ACE. What is the ratio of the area of the triangle to the area of the hexagon?

a) 1:2 b) 1:3 c) 2:3 d) $1:\sqrt{2}$ e) $1:\sqrt{3}$

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4. The value of $\tan^{-1}\left(2\sin\left(\frac{2\pi}{3}\right)\right)$ is given by NOTE: Some books use arctan for $\tan^{-1}$.

a) $\frac{\pi}{6}$ b) $\frac{\pi}{3}$ c) $\frac{2\pi}{3}$ d) $\frac{5\pi}{6}$ e) None of a) through d) is correct.

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5. A fair coin is tossed repeatedly. What is the probability that we obtain a total of two tails before we obtain a total of three heads?

a) 1/2 b) 9/16 c) 5/8 d) 11/16 e) 3/4

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6. Jill rides her bike around a course in the shape of an equilateral triangle. Her speed is 10 miles per hour on the first side of the course, 15 miles per hour on the second side of the course, and 20 miles per hours on the third and final side of the course. Then Jill's average speed during her ride

a) is less than 13 miles per hour.

b) is at least 13 but less then 14 miles per hour.

c) is at least 14 but less than 15 miles per hour.

d) is at least 15 miles per hour.

e) cannot be determined without more information.

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7. The real numbers x and y satisfy the equations $xy = \sin(2t)$ and $\frac{x}{y} = \tan(t)$ where $0 < t < \frac{\pi}{2}$. What is the value of $x^2 + y^2$?

a) $\sqrt{2}$ b) 1 c) 2 d) 4

e) The value cannot be determined uniquely from the given information.

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8. What is the surface area of the portion of the cylinder $x^2 + y^2 = 9$ that lies on or above the xy-plane and on or below the plane $x + 2z = 8$?

a) 15π b) 18π c) 21π d) 24π e) 27π

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9. Adam, his younger brother, Charlie, and his sister, Beth, have been saving their money (in U.S. currency) to buy their mother a special Mother's Day present. They discover that the gift can be purchased by using all the money saved by Charlie along with half that saved by Adam and Beth together, or by using all the money saved by Beth along with a third of that saved by Adam and Charlie together, or by using all the money saved by Adam along with a fourth of that saved by Beth and Charlie together. If the price of the gift is under one hundred dollars, what is the most it can cost?

a) \$85.34 b) \$89.68 c) \$92.17 d) \$99.96 e) \$99.99

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10. Which of the following statements correctly describes the graph of the equation $x^4 + 1 = 2x^2 + y^2$?

a) The graph is a pair of intersecting lines.

b) The graph is a pair of intersecting circles.

c) The graph is a pair of intersecting parabolas.

d) The graph is a pair of intersecting (noncircular) ellipses.

e) The graph is the union of an ellipse and a nonintersecting hyperbola.

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11. How many distinct real numbers belong to the following collection

$$\left\{ \ln\left(4-\sqrt{15}\right), \ln\left(4+\sqrt{15}\right), -\ln\left(4-\sqrt{15}\right), -\ln\left(4+\sqrt{15}\right), \ln\left(\frac{4+\sqrt{15}}{4-\sqrt{15}}\right), \ln\left(31+8\sqrt{15}\right) \right\}$$

a) 2 b) 3 c) 4 d) 5 e) 6

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12. The arithmetic mean (i.e the average) of N real numbers is N. The arithmetic mean of a subset of M of the given numbers is M, where $M < N$. What is the arithmetic mean of the remaining $N - M$ numbers?

a) M b) N c) $N - M$ d) $N + M$

e) The mean cannot be determined uniquely.

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13. Circle C_1 has center C, diameter AE, and a radius CG that is perpendicular to AE. Circle C_2 is internally tangent to C_1 at point E, intersects segment CA at point B, and intersects segment CG at point F. The length of segment AB is 10 and the length of segment GF is 6. What is the area of the crescent shaped region that is outside C_2 but inside C_1?

a) 105π b) 122π c) 127π d) 137π e) 155π

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14. Suppose that a and b are the two (complex) roots of $x^2 + 3x + 5 = 0$. If $t = \frac{a+2}{b+2}$ and $s = \frac{b+2}{a+2}$ are the two roots of $x^2 - mx + 1 = 0$, what is the value of m?

a) –5/3 b) –5/2 c) 5/2 d) 5/3 e) None of a) through d) is correct.

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15. With probability 1/2, each of the vertices of a cube is painted either red or black, and the colors are assigned independently. What is the probability that each pair of adjacent vertices will have different colors?

a) 0 b) 1/256 c) 1/128 d) 1/64 e) Greater than 1/64

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16. Consider the three by three matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & x \\ 4 & 9 & y \end{bmatrix}$. How many ordered pairs of positive integers (x,y) are there such that the determinant of this matrix is 0?

a) None b) Exactly two c) More than two but finitely many
d) Infinitely many but not all such pairs e) All such pairs

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17. If F is a function from the real numbers to the real numbers, we say that F has property (I) if $F(F(x)) = x$ for every real number x. We say that F has property (J) if $F(-F(x)) = -x$ for every real number x. Suppose that g is a function that satisfies property (I). How many of the following three functions satisfy property (J): $-g(-x)$, $-g(x)$ and $g(-x)$?

a) None b) Exactly one c) Exactly two d) All three
e) The answer cannot be determined without more information about g.

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18. If a is not zero and the three roots of $ax^3 + bx^2 + (a-4)x + 2 = 0$ are equal integers, with r the common value of these integers, what is the value of r?

a) 1 b) 2 c) 3 d) 4
e) The value cannot be determined uniquely from the given information.

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19. For each positive integer n, there is a jar that holds exactly n pennies and exactly $n + 2$ dimes. How many of these jars have the property that if two coins are removed at random and without replacement from the jar, the probability that at least one of the removed coins is a penny is 3/4 or more?

a) None of the jars has this property.

b) There is exactly one jar with this property.

c) There is more than one but only finitely many jars with this property.

d) Infinitely many, but not all, jars have this property.

e) Every jar has this property.

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20. $ABCD$ is a square with area 1. An equilateral triangle is inscribed in the square. One of the vertices of the equilateral triangle is A, another vertex lies on side BC and the third vertex lies on side CD. What is the perimeter of this equilateral triangle?

a) 3　b) $3\sqrt{6}-\sqrt{18}$　c) $1+3\left(\sqrt{6}-\sqrt{3}\right)$　d) $3\sqrt{(3/2)}$　e) $3\sqrt{2}$

PART II: 10 INTEGER ANSWER PROBLEMS

1. Consider the set of points in the xy-plane $\{(x,y): x=1,2,3,4,5;\ y=1,2,3,4,5\}$. If a and b are points in this set, let $D(a,b)$ be the distance between a and b. How many different positive values does $D(a,b)$ take on?

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2. Half the books on a teacher's bookshelf are mathematics books, a third of them are physics books, and 1/15-th of them are history books. The remainder of the books are romance novels. If 2 of the mathematics books, and 4 of the physics books are replaced by romance novels, then romance novels will comprise 15% of the books on the bookshelf. How many books, total, are there on the bookshelf?

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3. A box contains six ribbons that are identical in all aspects but their color. Two of the ribbons are red, and there is one orange ribbon, one yellow ribbon, one blue ribbon and one violet ribbon. Five different students choose, one after another, one ribbon from the box, randomly and without replacement. How many different distributions of colors of ribbons are possible?

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4. A bookkeeper at a car dealership is promised a car, a computer, and $18,000 in cash for a year's work at the dealership, with one-twelfth of the total value of the car, computer and cash earned each month that she works. She quits after seven months and is given the car and $5000 in cash. Had she worked another month, she would have received $1000 more in cash and the computer in addition to the car. To the nearest dollar, what is the value of the car?

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5. How many composite (i.e. non-prime) integers from 1 to 2009 have a prime number of (positive integral) divisors? [NOTE: 1 is <u>not</u> a prime.]

6. Suppose that a, b, c and d are non-negative integers and that the value of the determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is 1. If $a+b=7$ and $c+d=9$, what is the value of $a+d$?

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7. What four-digit integer n has the property that the value of $9n$ is the four-digit integer obtained by writing the digits of n in reverse order?

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8. A rectangle is inscribed in a triangle whose sides have lengths 30, 40 and 50. One edge of the rectangle lies on the hypotenuse of the triangle. What is the largest possible area the rectangle can have?

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9. Let z_j for $1 \le j \le 5$ denote the five distinct fifth roots of 1 in the complex plane. Let $w_j = \frac{z_j}{1+z_j^{\ 2}} + \frac{z_j^{\ 2}}{1+z_j^{\ 4}} + \frac{z_j^{\ 3}}{1+z_j} + \frac{z_j^{\ 4}}{1+z_j^{\ 3}}$ for $1 \le j \le 5$. How many distinct values belong to the collection $\{w_j : 1 \le j \le 5\}$?

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10. The sequence 4, 8, 13, 17, 22 . . . consists of the positive integers (in order) with the property that the sum of their digits is divisible by 4. What is the 2009th term in the sequence?

The following problem, which extends Integer Answer Problem #1, will be used only as part of a tie-breaking procedure. Do not work on it until you have completed the rest of the test.

TIE BREAKER PROBLEM

Consider the set of points in the xy-plane $\{(x,y): x = 1,2,\ldots,16;\ y = 1,2,\ldots,16\}$.
If a and b are points in this set, let $D(a,b)$ be the distance between a and b. How many different positive values does $D(a,b)$ take on?