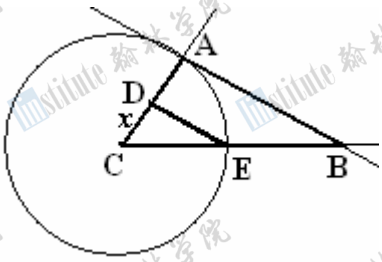
State Mathematics Finals: Geometry Solutions
May 3, 2007

1. b. The volume of a rectangular solid is $l \times w \times w = 7 \times 3 \times 5 = 105 cm^3$.

2. e. Since $\angle DEC \cong \angle ABC$, it follows that $\triangle DEC \approx \triangle ABC$, so



$$\frac{1}{x} = \frac{BE+1}{1} \Rightarrow x(BE+1) = 1 \Rightarrow$$

$$BE + 1 = \frac{1}{x} \Rightarrow BE = \frac{1}{x} - 1 = \frac{1-x}{x}$$

3. a. The sum of the measures of all the angles is 360 degrees. If we let the smallest angle have measure *x*, then $x + x + x + 3x = 360 \Rightarrow 6x = 360 \Rightarrow x = 60$, so the measure of the smallest angle is 60 degrees. But this makes the largest angle a straight angle, and this is a contradiction. So we need to let the largest angle be *x* and we get the equation, $x + x + x + \frac{x}{3} = 360 \Rightarrow \frac{10x}{3} = 360 \Rightarrow 10x = 1080 \Rightarrow x = 108$ and the smallest angle is 36 degrees.

4. a. There are seven digits, but only four distinct digits. The different permutations of these digits is $\frac{7!}{1!1!2!3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{2 \cdot 3!} = 7 \cdot 6 \cdot 5 \cdot 2 = 420.$

5. c. A normal year has 365 days, but a leap year has 366. 2008, 2012 and 2016 will be leap years, so there are $7(365) + 3(366) = 7(365) + 1098$ days between the two given dates. Since $1098 = 7(156) + 6$, the sum is one day short of 521 weeks, making the 2017 day a Wednesday.

6. d. Let the legs have length *a* and *b*. Since the area is $\frac{1}{2}ab = 10 \Rightarrow b = \frac{20}{a}$, so the hypotenuse will have length

$$\sqrt{a^2 + b^2} = \sqrt{a^2 + \left(\tfrac{20}{a}\right)^2} = \sqrt{\frac{a^4 + 400}{a^2}} = \frac{\sqrt{400 + a^4}}{a}.$$

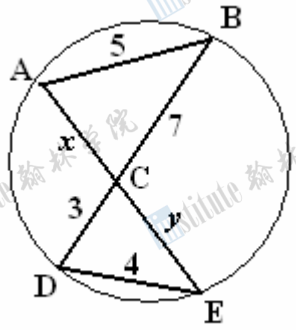
7. e. By larger, we are referring to area. The area of the outside ring is $\frac{\pi}{4}\left(5^2 - 3^2\right) = 4\pi$, while the area of the inside circle is $\frac{\pi}{4}1^2 = \frac{\pi}{4}$, so the outside ring is 16 times larger than the inner circle.

8. c. Since the cube is inside the sphere, the diameter of the sphere must equal the length of the diagonal of the cube. Let the length of the side be *s*, so the length of the diagonal is $s\sqrt{3}$. Now the volume of the sphere is

$\frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{s\sqrt{3}}{2}\right)^3 = \frac{\pi\sqrt{3}s^3}{2}$. The volume of the cube is simply s^3, so the ratio of the areas is $\frac{\pi\sqrt{3}s^3}{2} \div s^3 = \frac{\sqrt{3}}{2}\pi$.

9. a. Suppose (a) is the one false statement. Then all three (b), (c) and (d) must be true. Assuming (e) is also true but (a) false, then we have that (c) must be false, which is a contradiction.

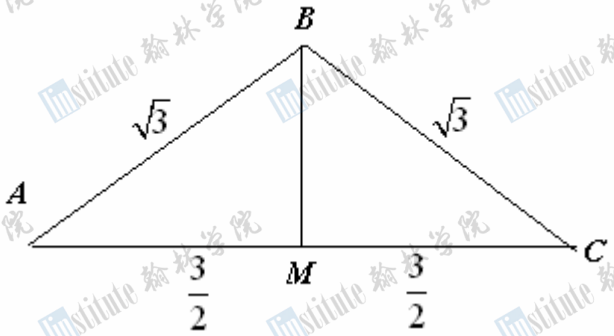
10. We know that $\triangle ABC \approx \triangle DEC$, so $\frac{AB}{DE} = \frac{5}{4}$, and $\frac{AB}{DE} = \frac{BC}{EC} = \frac{AC}{DC}$, so $\frac{5}{4} = \frac{7}{y} = \frac{x}{3} \Rightarrow 5y = 28, 4x = 15$, so $y = 5.6, x = 3.75$, so $AE = 5.6 + 3.75 = 9.35$.



11. c. If we let the length, width and height of the solid be *l*, *w*, and *h*, then the surface area will be $2(lw + lh + hw) = 52 \Rightarrow lw + lh + hw = 26$. But since the height is 2, we have $lw + 2l + 2w = 26$. If we solve this for one of the variables, and recall that all dimensions must be integers, we get $l = \frac{26 - 2w}{w + 2}$. Now since the numerator will always be even, that means *w* has to be even, which further means that the numerator can only by 22, 18, 14, 10, 6, or 2 for values of $w = 2, 4, 6, 8, 10,$ or 12. These combinations would make the length $\frac{22}{4}, \frac{18}{6}, \frac{14}{8}, \frac{10}{10}, \frac{6}{12},$ or $\frac{2}{14}$. There are two possible values then for the length, either 3 or 1. If the length is 3, the width is 4, and the height 2, so the volume is $3 \times 4 \times 2 = 24$. If the length is 1, the width would be 8 and the height 2, for a volume of 16, but that was not a choice.

12. c. Let the length and width be represented by *l* and *w*. The $l - 10 = w + 6$ and $(l - 10)(w + 6) = lw$. Let's get this into one variable, by substitution. First $l = w + 16$, then $(w + 6)(w + 6) = (w + 16)w \Rightarrow w^2 + 12w + 36 = w^2 + 16w$, so $4w = 36 \Rightarrow w = 9$. Now with $w = 9, l = 25,$ and the original volume is 225.

13. b. Since this is an isosceles triangle, the median BM will divide the triangle into two congruent triangles and by the Pythagorean



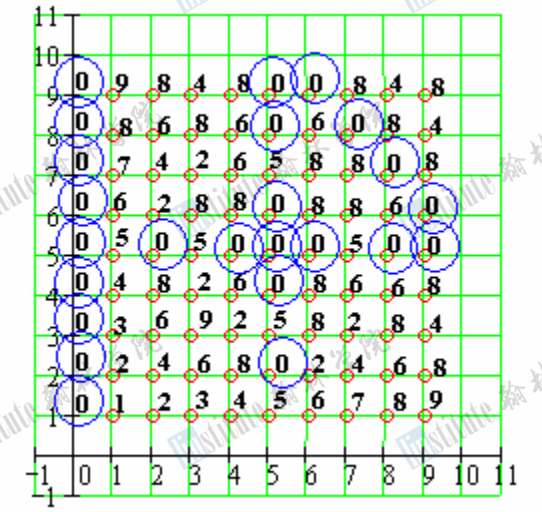
$$BM^2 = \left(\sqrt{3}\right)^2 - \left(\frac{3}{2}\right)^2 =$$

Theorem,

$$3 - \frac{9}{4} = \frac{3}{4}$$

So, $BM = \frac{\sqrt{3}}{2}$, making the $\tan\left(\angle C\right) = \frac{\sqrt{3}}{3}$. With a calculator (or this is actually an angle you should know, we find that this makes the angle exactly 30 degrees.

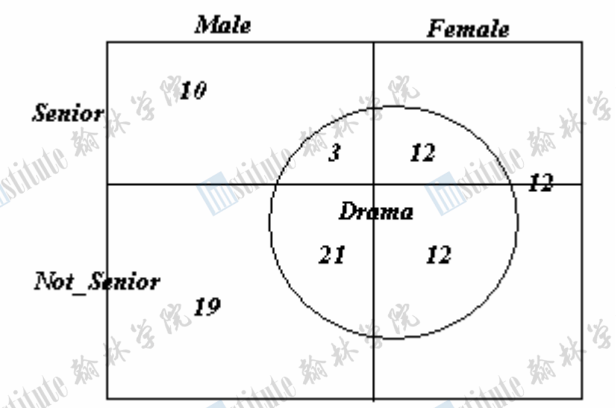
14. e. In the figure, let the horizontal coordinates (x) be the units digit and the vertical coordinates (y). The 90 small circles represent the numbers from 10 to 99. Next to each number is the final iteration when the function *P* defined in the problem is applied to the two digits in each number. Counting we see that there are 24 numbers that result in zero. $\frac{24}{90} = 26\frac{2}{3}\%$.



15. b. Let the lengths of the legs be *a* and *b*. Then $a^2 + b^2 = 25^2$ and $a + b + 25 = 56$. So $a + b = 31 \Rightarrow a = 31 - b$ and $\left(31 - b\right)^2 + b^2 = 25^2 \Rightarrow 961 - 62b + 2b^2 = 625$. This simplifies to $b^2 - 31b + 168 = 0 \Leftrightarrow \left(b - 7\right)\left(b - 24\right) = 0$, making *b* either 7 or 24. By symmetry, *a* would also be 7 or 24, making the only area $\frac{1}{2}\left(7\right)\left(24\right) = 84$

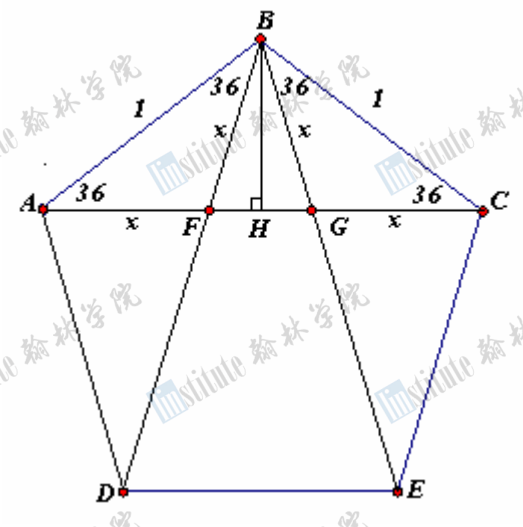
16. b. Let *F* be the number of faces, *E* the number of edges, and *V* the number of vertices. Euler's Formula states that $V + F - E = 2$, so $18 + F - 32 = 2 \Rightarrow F = 16$.

17. a. A quick Venn diagram can help place the students in their proper groups. The last statement places 15 drama students in the drama circle among the seniors and further puts 20% (3) of them as males and 12 as females. Then there must be 10 other male seniors by the second statement. The first statement tells us that there are 36 female students and



one-third (12) of them are drama majors but are not seniors. Now by the fourth statement, there must be 24 male drama majors, placing 21 in the non-senior sector. Finally, since there are 41 non-drama majors, there remaining 19 are non-seniors. Now adding all the non-overlapping sections gives 89 total students.

18. b. Thirty-six degrees is one of those special angles that pop up from time to time. To find the lengths of side in a triangle involving this angle, first construct draw a regular pentagon and construct 3 of the diagonals. Since each angle of the pentagon is 108 degrees, we have an isosceles triangle *ABC* with vertex angle of 108 degrees and base angles each of 36 degrees. This makes the base angles both 72 degrees. Now drop a perpendicular from B to diagonal $\overline{AC}$. Now $\triangle BCF$ is isosceles with $CF = 1$ and $FG = 1 - x$. Also we have $\triangle FBG \approx \triangle BCF$.



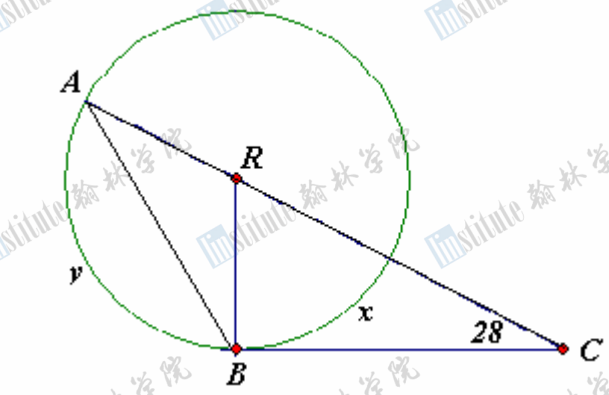
Now $\frac{BF}{FG} = \frac{BC}{BF} \Rightarrow \frac{x}{1-x} = \frac{1}{x} \Rightarrow x^2 = 1 - x$, so $x^2 + x - 1 = 0 \Rightarrow x = \frac{-1+\sqrt{5}}{2}$. Now in right triangle *BHC*, we have $BH^2 + HC^2 = 1$, but $HC = \frac{1}{2}(x+1) = \frac{1+\sqrt{5}}{4}$. So in a $36^o - 54^o - 90^o$ triangle, the ratio hypotenuse to the longest leg is $\frac{4}{1+\sqrt{5}}$ and in the triangle in the problem we this ratio is $\frac{?}{\sqrt{5}} = \frac{4}{1+\sqrt{5}} \Rightarrow ? = \frac{4\sqrt{5}}{1+\sqrt{5}} = 5 - \sqrt{5}$.

19. d. The circumference of a circle with 14" diameter is $C = 14\pi$. One-twelfth of this is $\frac{14\pi}{12} = \frac{7\pi}{6}$. Add in the two straight sides, each with length 7" and we get a total of $14 + \frac{7\pi}{6}$.

20. e The volume of the skin and the inside are equal, so we have $\frac{4}{3}\pi\left[(r+1.2)^3 - r^3\right] = \frac{4}{3}\pi r^3$. Simplify this to $r^3 - 3.6r^2 - 4.32r - 1.728 = 0$. Now using either the graphing (numerical) or algebraic solution feature of your calculator, you find that the only real solution is $r \approx 4.617$. Thus the diameter of the original orange is $2(4.617 + 1.2) = 2(5.817) = 11.634$.

21. b. The original cube has $l\times w\times h-\left[(l-2)\times(w-2)\times(h-2)\right]$ unit cubes on its surface. With the given lengths, we have $10\cdot12\cdot4-(8\cdot10\cdot2)=320$ cubes on the surface. Of these the 8 corners and 12 edges are painted with 3 and 2 faces black, respectively. So we have $320-8-4(8+10+2)=232$ with only one side painted black. This is $\frac{232}{480}=\frac{29}{60}$ of the total number of unit cubes.

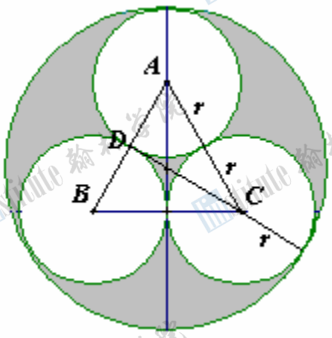
22. e. An angle formed by a secant and tangent to a circle has measure one-half the difference of the measure of the two arcs. So $28=\frac{1}{2}(y-x)\Leftrightarrow y-x=56$. But we also know that $y+x=180$, so solving this system gives us $y=118^o$ and $x=62^o$. An angle like $\angle BAC$, with vertex A on the circle is measured by one-half the arc tended, so its measure is 31 degrees.



23. d. When a number ending is 7 is raised to successive powers, the unit digit cycles through 7, 9, 3, 1, 7, 9, 3, 1, … in groups of 4. Thus the remainder when divided by 5 will cycle through 2, 4, 3, 1, 2, 4, 3, 1, …. Since $2007=4(501)+3$, the remainder will be 3.

24. e. If we consider one face of the pyramid, the base is 4 cm and the two equal sides have length 5. The altitude of this triangle will be $\sqrt{5^2-2^2}=\sqrt{21}$. Now using the segment from the apex of the pyramid as one leg of a right triangle, the altitude we just found would be the hypotenuse and the other leg measures 2, so the height of the pyramid is $\sqrt{\sqrt{21}^2-2^2}=\sqrt{21-4}=\sqrt{17}$.

25. a. We need to find the radii of the two circles to compute and subtract areas. If we let the area of the larger circle be 1 unit, then we can find the radii of the three smaller circle as follows. Notice that $\triangle ABC$ is equilateral. Segment $\overline{CD}$ bisects $\angle ACB$ and forms the longest leg of the 30-60-90 triangle $\triangle ACD$, so its length is $r\sqrt{3}$. The center of the circle is two-thirds of the way from C to D, so distance from the center of the large circle to C is $\frac{2}{3}r\sqrt{3}$. When the radius which is the extension of $\overline{DC}$ is added to this we get



$\frac{2}{3}r\sqrt{3}+r=r\left(\frac{2\sqrt{3}+3}{3}\right)$, but this is the radius of the larger circle, so $r\left(\frac{2\sqrt{3}+3}{3}\right)=1\Rightarrow r=\frac{3}{2\sqrt{3}+3}=2\sqrt{3}-3$. Now, the area of the shaded region is $\frac{\pi 1^2-3\pi\left(2\sqrt{3}-3\right)^2}{\pi 1^2}=\frac{36\sqrt{3}-62}{1}$.

26. b Since 132 rotations would be $132\frac{rev}{\min}\cdot 2\cdot\pi\cdot 45\frac{cm}{rev}\approx 37322.1\frac{cm}{\min}$. Since we want km/hr, we need to convert: $37322.1\frac{cm}{\min}\cdot 60\frac{\min}{hr}\cdot\frac{1km}{100000}\approx 22.4\frac{km}{hr}$

27. b. The interior angles of a pentagon must add up to $(5-2)180=540$ degrees. So we have $x+(x+20)+(x+40)+(x+60)+(x+80)=540$, so $5x+200=540\Rightarrow x=68$. So the largest angle is $68+80=148^o$.

28. d. If we let the lengths of the legs be *x* and *y*, then we know that $x^2+y^2=h^2$. We also know that the area, which is $\frac{1}{2}xy=36\Leftrightarrow xy=72\Leftrightarrow 2xy=144$. So if we subtract these we get $x^2-2xy+y^2=h^2-144$. Now factor to see that $(x-y)^2=h^2-144$, but since $(x-y)^2\geq 0$, we know that $h^2-144\geq 0\Rightarrow h\geq 12$.

29. e. Since $n=2k^2+1$, it follows that *n* must be odd. Since $nk=\left(2k^2+1\right)k=2k^3+k$, we can use induction to prove that such numbers are always divisible by 3. For $k=1$, we have $nk=\left(2\cdot 1^2+1\right)1=3$, which is divisible by 3. Now assume that this statement is true for any *k* and we will show that it will then have to be true for $k+1$. Assume $\left(2k^2+1\right)k$ is divisible by 3, this means that $\left(2k^2+1\right)k=2k^3+k=3m$ for some *m*. Now $\left(2(k+1)^2+1\right)(k+1)=\left(2k^2+4k+3\right)(k+1)=2k^3+6k^2+7k+3=\left(2k^3+k\right)+\left(6k^2+6k+3\right)$. Since we assumed that $2k^3+k$ was divisible by 3, all we need to show is that $6k^2+6k+3$ is divisible by 3. To show this, simply factor $6k^2+6k+3=3\left(2k^2+2k+1\right)$. Finally, we show that *n* is never divisible by 5. This one is a little tougher. If we look at the first few numbers generated this way (this is easy to do with the Table feature on most calculators), we see that these values are 3, 9, 19, 33, 51, 73, 99, 129, 163, 201, …. Now look at the remainders when dividing by 5. They are 3, 4, 4, 3, 1, 3, 4, 4, 3, 1, … . If we can show that this pattern must follow, then $2k^2+1$ will never by divisible by 5. Now assume that is not divisible by 5 and use this to show that

$2(k+5)^2+1=2k^2+20k+51=(2k^2+1)+(20k+50)$ has the same remainder when divided by 5. This is clearly true since the $20k+50$ has no remainder when divided by 5.

30. d. Since the top two rectangles have the same height, their lengths are proportional to their areas. So the top right rectangle is $\frac{21}{15}=\frac{7}{5}$ of the one on the left. The same will be true for the ones on the bottom, so the bottom right rectangle is $\frac{7}{5}\cdot 32=\frac{224}{5}=44.8$. So the area of the entire rectangle is $15+21+32+44.8=112.8$.

31. c. Let the sides be $5x, 4x, 3x$, so the volume is $60x^2$ while the surface area is $2(5x\cdot 4x+5x\cdot 3x+4x\cdot 3x)=94x^2$. If the ratio of the volume to surface area is 2:1, then $\frac{2}{1}=\frac{60x^3}{94x^2}\Leftrightarrow 188x^2=60x^3\Rightarrow x=\frac{188}{60}\Rightarrow 3x=3\left(\frac{188}{60}\right)=\frac{188}{20}=9.4$

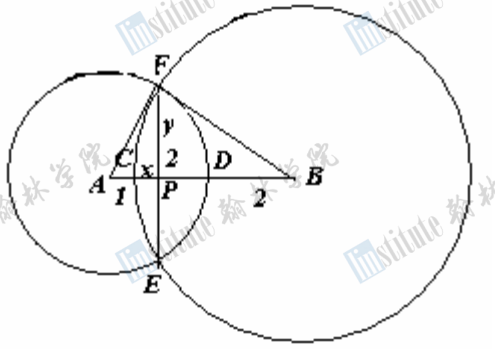
32. a. It does not matter which role is selected first. If the role of Romeo is filled first, the probability of filling with a male is $\frac{4}{6}$, followed by the role of Juliet at $\frac{2}{5}$. The product of these two is $\frac{4}{6}\cdot\frac{2}{5}=\frac{4}{15}$. If Juliet's role is filled first we have $\frac{2}{6}\cdot\frac{4}{5}=\frac{4}{15}$.

33. c. Since the points are evenly spaced, the arch subtended by the segment joining successive points will be $\frac{360}{2k+1}$ degrees. The desired inscribed angle will have half this measure, so it will be $\frac{180}{2k+1}$ degrees.

34. d. We want to use the largest digits in the hundreds and thousands place. So we will have either $600\cdot 7000$ or $700\cdot 6000$, but since both of these are the same, we need to place the next two digits. We will do this by using the next largest digit in the second position of the smaller number (so that it will be multiplied by the larger first digit of the other number. Using this strategy, we see that $750\cdot 6400=4,800,000<650\cdot 7400=4,810,000$. But we also see that $740\cdot 6500$ also equals 4,810,000, so we need to check the next digit. The four possibilities are $652\cdot 7430=743\cdot 652=4,844,360<653\cdot 7420=6530\cdot 742=4,845,260$. Now we

would like one more 742 instead of one more 653, so the largest product is $6531 \cdot 742 = 4,846,002$.

35. e. Draw $\overline{AF}$ and $\overline{BF}$ forming right triangles AFP and BFP. Let $FP = y, CP = x$. Now we have $FB^2 = FP^2 + BP^2$, so $4^2 = y^2 + (4-x)^2$ and $AF^2 = FP^2 + AP^2$ so $3^2 = y^2 + (1+x)^2$. When we solve this system of equations we see that $x = \frac{4}{5}$ and $y = \frac{12}{5} \Rightarrow FE = \frac{24}{5} = 4.8$.

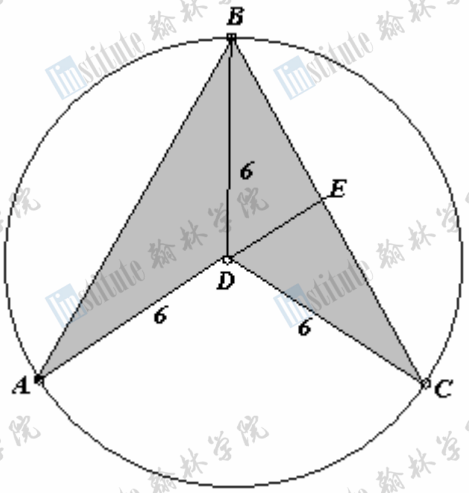


36. a. Fortunately for us, the given side form a right triangle. This means that the hypotenuse is also the diameter of the circle. Thus the area is $\pi r^2 = \pi\left(\frac{29}{2}\right)^2 = \frac{841\pi}{4}$.

37. d. Since $210 = 2 \cdot 3 \cdot 5 \cdot 7$, we can form factors by using each factor either one time or no time. Thus there are $2^4 = 16$ factors.

38. b. Let the radii of the two smaller circles be $2y$ and $3y$ respectively. This makes the radius of the larger circle $5y$. We are told that the area is 42π, so we have $42\pi = \frac{1}{2}\left(\pi(5y)^2 - \pi(3y)^2 - \pi(2y)^2\right)$, so $84\pi = \left(25\pi y^2 - 9\pi y^2 - 4\pi y^2\right) = 12\pi y^2 \Rightarrow y^2 = 7 \Rightarrow y = \sqrt{7}$. But $x = 5y \Rightarrow x = 5\sqrt{7}$.

39. a. The shaded region is made up of two isosceles triangles. Each of them can be divided into two 30-60-90 right triangles. In $\triangle CDE$, we have $DE = 3, CE = 3\sqrt{3}$. So the area of this triangle is $\frac{1}{2} \cdot 3 \cdot 3\sqrt{3} = \frac{9\sqrt{3}}{2}$. There are 4 triangles so the total area is $4\left(\frac{9\sqrt{3}}{2}\right) = 18\sqrt{3}$



40. e. To lower the sign 3 feet, you must

unwrap 9 feet from the spool. Each complete turn of the crank will only turn the spool one-third of a revolution. The circumference of the center of the spool is 2π, so the spool must turn $\frac{9}{2\pi}$ turns, and the crank must turn $\frac{9}{2\pi} \cdot 3 = \frac{27}{2\pi} \approx 4.297$ turns. Since there is only one answer with this magnitude, the direction is irrelevant. However, the circles will turn in opposite directions, so the spool must turn clockwise to let out the rope and the crank has to turn counter clockwise.