NC STATE MATHEMATICS CONTEST – APRIL 2007: SOLUTIONS

1. Answer **d**): We want $a^2/4 = a$ so $a = 4$.

2. Answer **c**): The areas of *ABE* and *AFD* each equal 1/4 of the area of *ABCD*, and the area of *ECF* is 1/8 of the area of *ABCD*. 1 – 1/2 – 1/8 = 3/8.

3. Answer **b**): The first marble falls a foot after 1/4 seconds. If we let $y = 0$ at the bottom of the cliff, $y_2(t) = 64 - 16(t - 1/4)^2$ and $y_2(2) = 15$

4. Answer **a**): (I) is true with side lengths $\{a,b,b\}$ if $a < b$. (II) is false when $2a < b$. (III) is false since $b\sqrt{a^2 - (b/2)^2}/2 = a\sqrt{b^2 - (a/2)^2}/2 \Rightarrow a = b$.

5. Answer **e**): Thinking about the intersection of the oil with a disk of radius 6 perpendicular to the axis of the cylinder, we see that the volume is $12\left[\dfrac{6^2\, 2\pi/3}{2} - \dfrac{6^2 \sin(2\pi/3)}{2}\right] = 265.33$

6. Answer **b**): Let $x = b^t$ and consider $\log_b(b^t) = \log_{b^t}(b)$ or $t = 1/t$. Then, $t = \pm 1$ and $x = b$ or $1/b$.

7. Answer **e**): Think about placing ten 5 by 4π rectangles side by side with the 5 inch edges adjacent. The path of the ant forms a straight line from the lower left hand corner of the leftmost rectangle to the upper right corner of the rightmost rectangle.

8. Answer **b**): The function $x^{1/x}$ increases up to its maximum at x = e and decreases thereafter. Also, $a^b = b^a \Rightarrow a^{1/a} = b^{1/b}$. Thus, if equation has a solution in integers, $a < b$, $a = 1$ or 2. We see that only $a = 2$ and $b = 4$ solve the given equation.

9. Answer **c**): $x + (1-y) > (y-x)$, $x + (y-x) > 1-y$, $(1-y) + (y-x) > x$, $0 < x < 1/3$, $2/3 < y < 1$ must all hold. Thus we need (x,y) in the square $0 < x < 1/3$, $2/3 < y < 1$ which satisfy $y < 1/2 + x$. These points lie in a right triangle with base and height 1/6, and (1/72)/(1/9) = 1/8.

10. Answer **d**): $p + n + d + q = 101$, $p + 5n + 10d + 25q = 582$, $p + n = 15q$, $p = d + q + 6$ give $p = 107 - 15q$, $n = 30q - 107$, $d = 101 - 16q$ so that q = 4, 5 or 6.

11. Answer **a**): $S(1) + S(2) + \cdots + S(999) = 3 \cdot 100 \cdot (0 + 1 + \cdots + 9) = 13{,}500$, $S(1000) + S(1001) + \cdots + S(1999) = 3 \cdot 100 \cdot (0 + 1 + \cdots + 9) + 1000 = 14{,}500$, and $S(2000) + S(2007) = 16 + 28 = 44$.

12. Answer **b**): The side lengths of the triangle are the sums of the pairs of the radii. Thus the semiperimeter of the triangle is 144 + 225 + 256 = 625. Applying Heron's Formula, the area is the square root of $625 \cdot 144 \cdot 225 \cdot 256$.

13. Answer **a**): Applying Ptolemy's Theorem to the cyclic quadrilateral *APBC*, $z|AB| = x|BC| + y|AC|$.

14. Answer **d**): $|2iz + 4| = 2|z - 2i|$ so $z = 3 + 2i$ gives the maximum.

15. Answer **b**): A given team wins both its games and is thus the unique winner of the tournament with probability 1/4. If no team wins both its games, the tournament ends in a three way tie.

OVER FOR SOLUTIONS TO THE REMAINING PROBLEMS.

16. Answer **d)**: Edges 2, 4 and 6 satisfy the given conditions. If there were an edge of length 7, the sum of the lengths of the other two edges would be 5 and the product of their lengths would be 48/7. However, if $x + y = 5$, xy is at most 25/4.

17. Answer **a)**: With $x = \sqrt{b}\cos(t)$, $y = \sqrt{a}\sin(t)$, $xy = \left(\sqrt{ab}\sin(2t)\right)/2$.

18. Answer **d)**: Let $P = ABCDE$, in clockwise order. Suppose that AD and BE meet at Q and AC and BE meet at R. Since $\angle DAC = \pi/5$, ΔABQ is isosceles with $|BQ| = 1$. Let $s = |QR|$ and $r = |RB|$; we want s. The Law of Sines gives $s/\sin(\pi/5) = r/\sin(2\pi/5) \Rightarrow r = \left(\frac{1+\sqrt{5}}{2}\right)s \Rightarrow s = \frac{3-\sqrt{5}}{2}$.

19. Answer **c)**: $\frac{1}{a_k\sqrt{a_{k+1}} + a_{k+1}\sqrt{a_k}} = \frac{1}{2}\left(\frac{1}{\sqrt{a_k}} - \frac{1}{\sqrt{a_{k+1}}}\right)$ so $S(n) = \frac{1}{2} - \frac{1}{2\sqrt{2n+1}}$, and this last equals 1003/2007 if n = 2,014,024.

20. Answer **e)**: Let $n = 2^k 5^m q$ where gcd(q,10) = 1. Choose r so that $10^r \equiv 1 \pmod{q}$, and let $x = 10^r$. Then $X = x^0 + x^1 + \dots + x^{q-1} \equiv 0 \pmod{q}$. Then $10^{\max(k,m)}X$ is a multiple of n of the desired form.

<u>Integer Answer Problems</u>

1. Answer: **91** We can assume $a + 35 = b$ and $ab = 7 \cdot 252$. It follows that a = 28 and b = 63.
2. Answer: **36,000** P(6,4)P(5,3) + P(6,3)P(5,4) = 36,000
3. Answer: **8** If n orange jellybeans are added, we want $1 - C(8,3)/C(8+n,3) \geq 9/10$.
4. Answer: **123** $\sum_{k=0}^{n}(n+k)(2n-k) = 2n^2(n+1) + n^2(n+1)/2 - n(n+1)(2n+1)/6 = n(n+1)(13n-1)/6$, and this exceeds the square of 2007 if $n > 122$.
5. Answer: **47** Let $\theta = \angle BAC$. Then $15 = 6 + |AC| + |BC| = 6 + 6\cos(\theta) + 3(2\theta) \Rightarrow \theta = 46.514°$.
6. Answer: **15** If only one color is used, there are 3 ways. If three faces are painted one color and one another color, there are 6 ways. If two faces are painted one color and the other two faces are painted a different color, there are 3 ways. If all three colors are used, there are 3 ways.
7. Answer: **99** $x = |BC| \Rightarrow n = \sqrt{100^2 + x^2} - x = 100^2/\left(\sqrt{100^2 + x^2} + x\right)$. This is a decreasing function of x with value 100 at $x = 0$. Thus there are 99 possible values for x. Each such x is a positive rational.
8. Answer: **15** $1/a + 1/b + 1/c = (ab + ac + bc)/(abc) = (-120/5)/(-8/5) = 15$.
9. Answer: **1673** $1 \leq a \leq 6 \Rightarrow a^6 \equiv 1 \pmod{7}$ and $1^n + 2^n + 3^n + 4^n + 5^n + 6^n$ is divisible by 7 for $1 \leq n \leq 5$ but not $n = 6$. Thus, $1^n + 2^n + 3^n + 4^n + 5^n + 6^n$ is divisible by 7 if n is not a multiple of 6.
10. Answer: **2** 1/6 + 2/8 + 3/9 + 5/4 = 2, and $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} \geq \frac{a+c+e+g}{9} \geq \frac{10}{9}$.

TIE BREAKER ANSWER: $\sum_{k=5}^{10} C(a,k)C(20,10-k)/C(20+a,10) > 0.95$ for $a >$ **41**.