NC STATE MATHEMATICS CONTEST – APRIL 2007

PART I: 20 MULTIPLE CHOICE PROBLEMS

1. For what positive real number a do the graphs of $y = x(a - x)$ and $y = a$ meet in exactly one point?

a) $a = 1/2$ b) $a = 1$ c) $a = 2$ d) $a = 4$ e) There is no such a.

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2. Suppose that $ABCD$ is a parallelogram, with the vertices listed in clockwise order, and that E is the midpoint of side BC and that F is the midpoint of side CD. What is the ratio of the area of the triangle AEF to the area of the entire parallelogram?

a) 1/8 b) 1/4 c) 3/8 d) 1/2

e) The ratio cannot be determined uniquely from the given information.

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3. According to the free-fall model, an object released from rest will fall $16t^2$ feet, t seconds after it is dropped. Suppose that two marbles are dropped, one after the other, from a cliff that is 64 feet high, and that the second marble is released the instant the first marble has fallen exactly one foot. According to the free-fall model, how far above the ground is the second marble at the instant the first marble strikes the ground?

a) 1 foot b) 15 feet c) 16 feet d) 32 feet e) more than 32 feet

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4. Consider the following three statements.

(I) If a and b are distinct positive real numbers, there is an isosceles triangle whose side lengths are either $\{a,a,b\}$ or $\{a,b,b\}$.

(II) If a and b are distinct positive real numbers, there is an isosceles triangle whose side lengths are $\{a,a,b\}$ and another isosceles triangle whose side lengths are $\{a,b,b\}$.

(III) There are distinct positive real numbers a and b such that there is an isosceles triangle whose side lengths are $\{a,a,b\}$ and another isosceles triangle whose side lengths are $\{a,b,b\}$, and these two triangles have the same area.

Which of (I), (II) and (III) are true?

a) Only (I) is true. b) Only (III) is true. c) Only (I) and (III) are true.

d) All three are true. e) None are true.

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5. An underground oil tank is a right circular cylinder of radius 6 ft and height 12 ft. The tank is buried horizontally with the axis of the cylinder parallel to the surface of the ground. If the depth of the oil in the tank is 3 ft (at the deepest point), what is the volume of oil in the tank to the nearest cubic foot?

a) 261 b) 262 c) 263 d) 264 e) 265

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6. If b is a positive real number different from 1, let $\log_b(x)$ denote the base b logarithm of x. Let N be the number of solutions x to the equation $\log_b(x) = \log_x(b)$ where x is a positive real number different from 1. Which of the following statements about N is correct?

a) $N = 1$ b) $N = 2$ c) $3 \le N \le 4$ d) N is greater than 4 but finite

e) N is infinite.

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7. A coffee can in the shape of a right-circular cylinder of height 5 inches and diameter 4 inches is sitting on a kitchen tabletop. An ant crawls with constant speed along the surface of the can in such a way that the path of the ant is a spiral that begins at the base of the can and circles the can exactly 10 times before reaching the top. If the height of the ant above the tabletop increases at a constant rate, what is the total distance traveled?

a) 40π b) $5+40\pi$ c) $50+40\pi$ d) $\sqrt{1600\pi^2+5}$ e) $5\sqrt{64\pi^2+1}$

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8. The number of ordered pairs (a,b) of positive integers such that $a < b$ and $a^b = b^a$ is

a) 0 b) exactly one c) exactly two

d) more than 2 but finite e) infinite

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9. A value of x is chosen randomly from the interval [0, 1/3]. Then a value of y is chosen randomly from the interval [2/3, 1]. What is the probability that there is a triangle whose edges have length x, $1 - y$, and $y - x$?

a) 0 b) 1/16 c) 1/8 d) 3/16 e) None of a) through d) is correct.

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10. A coin collection consists of pennies, nickels, dimes and quarters. The collection contains a total of 101 coins, the total value of the 101 coins is $5.82, and the number of pennies and nickels combined is 15 times as large as the number of quarters. Finally, the number of dimes and quarters combined is 6 less than the number of pennies. Which of the following statements about the number of such collections is correct if we consider two collections to be different if the number of coins of even one denomination differs from one collection to the other?

a) No such collection can exist. b) There is exactly one such collection.

c) There are exactly two such collections. d) There are exactly three such collections.

e) There are more than three such collections.

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11. If n is a positive integer, let $S(n)$ denote the sum of the base 10 digits of n. What is the sum of $S(1)+S(2)+\cdots+S(2007)$?

a) 28044 b) 28045 c) 28046 d) 28047 e) None of a) through d) is correct.

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12. A circle is constructed at each vertex of a triangle in such a way that each of the circles is tangent to the other two circles. The lengths of the radii of the circles are 144, 225 and 256. What is the area of the triangle?

a) 14,400 b) 72,000 c) 360,000

d) The area cannot be determined uniquely from the given data.

e) None of a) through d) is correct.

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13. A circle is circumscribed about an equilateral triangle, *ABC*, and a point, *P*, on the (shorter) arc joining *A* and *B*, is chosen. Let $x = |PA|$, $y = |PB|$, $z = |PC|$. Note that *z* is larger than both *x* and *y*. Which of the following statements is correct?

a) For every *P*, $x + y$ must be equal to *z*.

b) For every *P*, $x + y$ must be greater than *z*.

c) For every *P*, $x + y$ must be less than *z*.

d) For some *P*, $x + y$ is greater than *z*, for other *P*, $x + y$ is less than *z*, but $x + y$ is never equal to *z*.

e) Each of the possibilities $x + y$ greater than *z*, equal to *z* and less than *z* is possible for some *P*.

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14. Suppose that *z* is a complex number that satisfies $|z - 2 - 2i| \le 1$ (where $i^2 = -1$ and $|w|$ denotes the modulus or absolute value of the complex number *w*). The maximum value of $|2iz + 4|$ is then

a) 4 b) $2\sqrt{5}$ c) $4\sqrt{2}$ d) 6 e) $4\sqrt{3}$

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15. Three teams participate in a tournament in which each team plays both of the other two teams exactly once. The teams are evenly matched so that in each game, each team has a 50% chance of winning the game. No game can end in a tie. At the end of the tournament, if one team has more wins than both of the other two teams, that team is declared the unique winner of the tournament. Otherwise, the tournament ends in a tie. What is the probability that the tournament ends in a tie.

a) 1/8 b) 1/4 c) 3/8 d) 1/2 e) 5/8

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16. A rectangular box has volume 48, and the sum of the lengths of the twelve edges of the box is 48. What is the largest integer that could be the length of an edge of the box?

a) 12 b) 10 c) 8 d) 6 e) None of a) through d) is correct.

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17. Suppose that a and b are positive numbers. If (x,y) is a point on the curve $ax^2 + by^2 = ab$, what is the largest possible value that xy can assume?

a) $\sqrt{ab}/2$ b) $\sqrt{ab}$ c) $\dfrac{ab}{a+b}$ d) $\dfrac{2ab}{a+b}$

e) xy has no maximum for points (x,y) on the curve.

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18. Let P be a regular pentagon with side length 1. The points, inside P, at which the diagonals of P meet form a smaller regular pentagon. The side length of this smaller pentagon can be written in the form $a + b\sqrt{5}$ where a and b are rational numbers. What is the sum of a and b?

a) $a + b = -2$ b) $a + b = -1$ c) $a + b = 0$ d) $a + b = 1$ e) $a + b = 2$

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19. Let $a_k = 2k - 1$ and let $S(n) = \sum_{k=1}^{n} \dfrac{1}{a_k\sqrt{a_{k+1}} + a_{k+1}\sqrt{a_k}}$. Let N be the smallest value of n such that $S(n) \geq \dfrac{1003}{2007}$. Which of the following statements about N is correct?

a) $N \leq 2007$ b) $2007 < N \leq 2007\sqrt{2007}$ c) $2007\sqrt{2007} < N \leq 2007^2$ d) $2007^2 < N$

e) N does not exist since there is no n such that $S(n) \geq \dfrac{1003}{2007}$.

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20. A positive integer n has property D if the decimal expansion of some positive integral multiple of n contains only 0's and 1's. Thus, 337 has property D since $3 \times 337 = 1011$. Which of the following statements about the set, S, of positive integers with property D is correct?

a) S is finite.

b) S contains infinitely many odd integers but only finitely many even integers.

c) S contains infinitely many even integers but only finitely many odd integers.

d) There are infinitely many even and infinitely many odd integers in S, but not every positive integer is in S.

e) Every positive integer is in S.

THIS CONCLUDES PART I. PART II BEGINS ON THE FOLLOWING PAGE.

PART II: 10 INTEGER ANSWER PROBLEMS

1. Suppose that a and b are positive integers whose greatest common divisor is 7 and whose least common multiple is 252. If $|b-a|=35$, what is the value of $a+b$?

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2. Let C be the set of six consonants {b, c, d, f, g, h} and let V be the set of five vowels {a, e, i, o, u}. Let W be the set of seven letter "words" that can be formed with these 11 letters using the following rules: (a) The vowels and consonants in the word must alternate. (b) No letter can be used more than once in a single word. How many words are in the set W if we assume that two words are the same if and only if they contain exactly the same letters in exactly the same sequence?

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3. A jar contains 8 green jellybeans, and we have an unlimited supply of orange jellybeans, some of which we will add to the jar. Once we have added the orange jellybeans, we intend to choose three jellybeans at random from the jar, and we want the probability that we obtain at least one orange jellybean to be at least 90%. What is the smallest number of orange jellybeans that we must add to the jar to achieve this goal?

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4. For each positive integer n, let $S_n=(n)(2n)+(n+1)(2n-1)+(n+2)(2n-2)+\cdots+(2n)(n)$. What is the smallest value of n such that S_n is greater than the square of 2007?

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5. Suppose that Γ is a circle with radius 3 and that A and B are the endpoints of a diameter of Γ. Suppose also that C is a point on Γ such that the sum of the lengths of the segments AB and AC and the length of the arc BC is 15. To the nearest degree what is the degree measure of $\angle CAB$?

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6. Suppose that you have a large supply of regular tetrahedra, all the same size, and three colors of paint, red, white, and blue. If each face of a tetrahedron is to be painted a single color, in how many different ways can you paint the tetrahedra? (Two tetrahedra have been painted "differently" provided it is impossible to position the tetrahedra in the same direction such that corresponding faces are painted the same color.)

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7. How many right triangles, ABC, with hypotenuse AB satisfy the following properties?

   (i) $|AC|$, the length of AC, is 100.

   (ii) $|AB|-|BC|$ is a positive integer.

   (iii) $|BC|$ is a positive rational number.

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8. If a, b and c are the zeros, possibly complex, of the polynomial $5x^3 + 1440x^2 - 120x + 8$, what is the absolute value of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$?

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9. For how many positive integers, n, less than 2008 is it the case that $1^n + 2^n + 3^n + 4^n + 5^n + 6^n$ is divisible by 7?

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10. Suppose that $\{a,b,c,d,e,f,g,h\} = \{1, 2, 3, 4, 5, 6, 8, 9\}$ and that $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = N$ where N is an integer. What is the smallest possible value for N? CAUTION: The integer 7 is **NOT** allowed as a numerator or a denominator in the sum $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h}$.

The following problem will be used only as part of a tie-breaking procedure.
You should not work on it until you have completed the rest of the test.

TIE BREAKER PROBLEM

Consider the situation described in Integer Answer problem 3, modified as follows. The jar initially contains 20 green jellybeans, and we intend to choose at random 10 jellybeans from the jar. We want the probability that we obtain at least 5 orange jellybeans to be at least 95%. What is the smallest number of orange jellybeans that we must add to the jar to achieve this goal?