2007 Algebra II Solutions
May 3, 2007

1. C If $p=\frac{1}{3}$, then $p>p^2$ since $\frac{1}{3}>\left(\frac{1}{3}\right)^2=\frac{1}{9}$, so * * is printed.

2. B 5B9 is divisible by 9 when B = 4. Then $2A3+326=549 \Rightarrow 2A3=223$ and $A=2$, So $A+B=2+4=6$.

3. A $5x+12y=60 \Rightarrow y=5-\frac{5}{12}x$, so $\sqrt{x^2+y^2}=\sqrt{x^2+\left(5-\frac{5}{12}x\right)^2}=\sqrt{\frac{169}{144}x^2+\frac{25}{6}x+25}$. A graphing calculator shows the minimum value of this expression is approximately 4.6153846. The closest rational choice to this approximation is $\frac{60}{13}$. To see that this is actually an approximation for a rational number, it is possible to re-express $\sqrt{\frac{169}{144}x^2+\frac{25}{6}x+25}=\sqrt{\left(\frac{13}{12}x+\frac{25}{13}\right)^2+\left(25-\left(\frac{25}{13}\right)^2\right)}$ $=\sqrt{\left(\frac{13}{12}x+\frac{25}{13}\right)^2+\frac{25\cdot169-25^2}{169}}=\sqrt{\left(\frac{13}{12}x+\frac{25}{13}\right)^2+\left(\frac{25\cdot144}{169}\right)}$. When the term involving the variable x is the expression is zero (as small as you can make it), the remaining term is $\frac{60}{13}$.

4. E Let A denote the number is attendance in Atlanta and let B denote the number is attendance in Boston. We are given $45,000 \le A \le 55,000$, and $0.9B \le 60,000 \le 1.1B$, so $54,546 \le B \le 66,666$. Hence the largest possible difference between A and B is $66,666-45,000=21,666$, so the correct choice is E.

5. D The least integer is 100 and the greatest is 1225 and their difference is 1125.

6. A Let x denote the number of days John did not work. Then he worked $A-x$ days and so earned $B(A-x)-Cx=D$ dollars. Solving this for x we get $-(B+C)x=D-AB \Rightarrow x=\frac{AB-D}{B+C}$.

7. A $12=(x+5)^2+8 \Rightarrow 4=(x+5)^2 \Rightarrow \pm2=(x+5)$, so $x=-3$ or $x=-7$. So $-3+(-7)=-10$.

8. D $f(2)=f(1+1)=\left(f(1)\right)^2=4$ and $f(3)=f(2+1)=\left(f(2)\right)^2=16$, and $f(4)=f(3+1)=\left(f(3)\right)^2=16^2=256$.

9. B Two lines with infinite solutions are coincident, so $\frac{b}{1}=\frac{a}{3}=\frac{5}{a}\Rightarrow a=\sqrt{15}, b=\frac{\sqrt{15}}{3}\approx 1.290$.

10. A Let n be the number of nickels, d the number of dimes, and q the number of quarters. Then
$$\left\{\begin{matrix} n+d+q=56 \\ 0.05n+0.10d+0.25q=9 \\ 0.25q=2\left(0.05n+0.10d\right)\end{matrix}\right\}\Rightarrow q=24, n=4.$$

11. B $2^x=3\Rightarrow x=\log_2 3$, so $3^x=3^{\log_2 3}\approx 5.7$

12. D Let x and y be the dimensions of the rectangle. Then
$2x^2+2y^2=100\Rightarrow x^2+y^2=50\Rightarrow\sqrt{x^2+x^2}=\sqrt{50}=5\sqrt{2}$ and the length of the diagonal is $\sqrt{x^2+y^2}$.

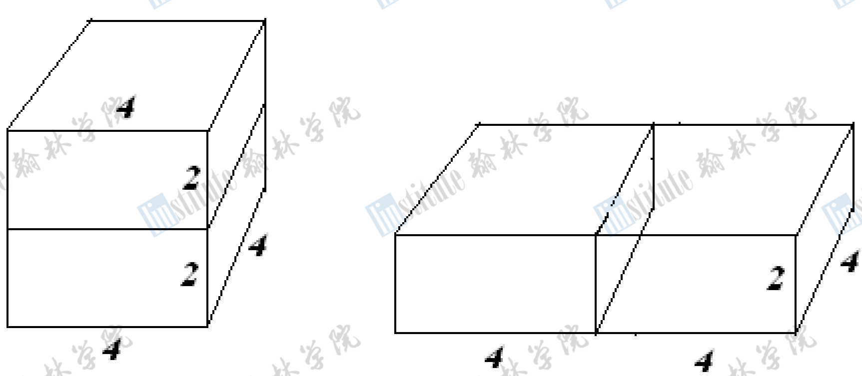
13. D $g(18)=4$, since 4 is the least integer greater than $\frac{18}{5}$. Similarly $f(102)=20$, since 20 is the greatest integer less than $\frac{102}{5}$. So $g(18)+g(20)=4+20=24$.

14. C Look beyond integers 1 and 3 and consider non-integer rational numbers such as $\frac{1}{3}$ for x. Then $\frac{x}{3}$ and x are not integers. Note $6x=2\cdot 3x$ so $6x$ must be an integer. Only III is an integer.

15. D Since 6a is even and 9 is odd $\frac{(6a)^*}{9^*}=\frac{3a}{3}=a$. Note $a\neq a^*$. Since 2a is even $(2a)^*=2a(0.5)=a$.

16. D There are 9 ordered triples, (0,0,2), (0,2,1), (5,1,1), (10,0,1), (0,4,0), (5,3,0), (10,2,0) (15,1,0) and (20,0,0).

17. B The volume of a cube is equal to an edge cubed, so $e^3=64$ and each edge has length 4. If the cube is sliced horizontally in two, each of the resulting solids will have two sides of length 4 and one of length 2. So when they are glued together, the resulting figure will have one edge of length 2, one of length 4, and one of length 8.



The surface area is the sum of the areas of the solid's six faces. The top and bottom of each have area $8\times4=32$, the front and back each have area $8\times2=16$, and each side has area $4\times2=8$. So the surface area of the new solid is $2(32)+2(16)+2(8)=64+32+16=112$.

18. B Since k is odd, $f(k)=k+3$. Since $k+3$ is even, $f(k+3)=f\left(f(k)\right)=\frac{k+3}{2}$. If $\frac{k+3}{2}$ is odd, then $27=f\left(f\left(f(k)\right)\right)=f\left(\frac{k+3}{2}\right)=\frac{k+3}{2}+3$, which implies that $k=45$. This is not possible because $f\left(\left(f(45)\right)\right)=f\left(f(48)\right)=f(24)=12$. Hence $\frac{k+3}{2}$ must be even, and $27=f\left(f\left(f(k)\right)\right)=f\left(\frac{k+3}{2}\right)=\frac{k+3}{4}$, which implies that $k=105$. Checking, we find that $f\left(\left(f(105)\right)\right)=f\left(f(108)\right)=f(54)=27$. Hence the sum of the digits of k is $1+0+5=6$.

19. C Since $E(100)=E(00)$, the result is the same as $E(00)+E(01)+E(02)+\cdots+E(99)$, which is the same as $E(00010203\cdots99)$. There are 200 digits, and each digit occurs 20 times. So the sum of the even digits is $20(0+2+4+6+8)=20(20)=400$.

20. D

21. B $29+D=30+B\Rightarrow D-B=1$. $14+B+D=30+B\Rightarrow D=16$. So $D-B=1\Rightarrow16-B=1\Rightarrow B=15$.

22 A $f\left(g(x)\right)=f\left(|2x+3|\right)=|2x+3|^2-1=\left|4x^2+12x+9\right|-1$ and $g\left(f(x)\right)=g\left(x^2-1\right)=\left|2\left(x^2-1\right)+3\right|=2x^2+1$. So $f\left(g(3)\right)=80, g\left(f(2)\right)=9,$ $g\left(f(-1)\right)=3, f\left(g(-5)\right)=48,$ and $g(10)=23,$ so $f\left(g(3)\right)$ has the largest value.

23. C The 11 such integers with no partners in the set are 5, 7, 11, 13, 14, 15, 17, 19, 21, 22, 23.

24. E $\frac{A}{2x-7}+\frac{B}{x+3}=\frac{A(x+3)+B(2x-7)}{(2x-7)(x+3)}=\frac{19x-8}{2x^2-x-21}$, so $A=9$ and $B=5$ and $A+B=14$.

25. C There are $\binom{8^2}{2}=\frac{8^2\left(8^2-1\right)}{2}$ ways of choosing two arbitrary squares for the defective tiles.

If the two defective tiles share and edge, then two cases must be considered.
Case1: One of the tiles was placed in any of the top 7 rows (8 x 7 ways), and the other was placed in the square below.

Case 2: One time was placed in any of the left 7 columns (8 x 7 ways), and the other was placed the square to its right.

So the probability is $\dfrac{2\times8\times7}{\left(\dfrac{64\times63}{2}\right)}=\dfrac{1}{18}$.

26. D ~ represents multiplication. Since $1001=7\cdot11\cdot13$, the 7th 11th and 13th letters are the solution. Those letters are G, K and M.

27. E $3^x=5\Rightarrow 3^{2x}=\left(3^x\right)^2=5^2=25$, so $3^{2x+3}=3^{2x}\cdot3^3=25\cdot27=675$.

28. B In one revolution, the original car tire covers $C=\pi d=\pi\left(25in\right)$. Since 1 mile equals $5280\cdot12=63360$ inches, the tire turns $\dfrac{63360\,\frac{in}{mi}}{25\pi\,\frac{in}{rev}}\approx 806.7245755\,\frac{rev}{mi}$. The tire with decreased radius has $r=12.25, d=24.5"$ and $C=24.5\pi"$. So $\dfrac{63360\,\frac{in}{mi}}{24.5\pi\,\frac{in}{rev}}\approx 823.1883424\,\frac{rev}{mi}$. The percent increase is $\dfrac{\dfrac{63360}{25\pi}-\dfrac{63360}{24.5\pi}}{\dfrac{63360}{25\pi}}\approx 2.0\%$.

29. B Since 5 + 2 = 7, and 3 + 4 = 7, each side length is a multiple of 7, so 5:2 can be represented as 5a:2a and 3:4 can be represented at 3a:4a. Then the area

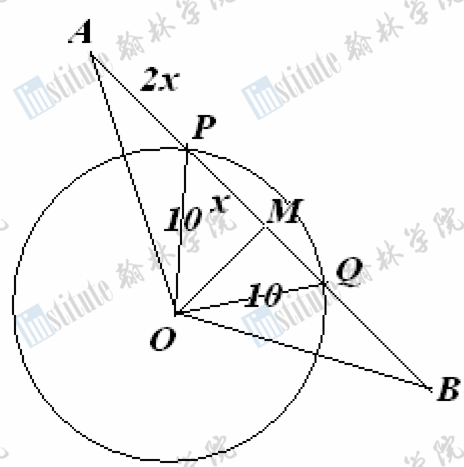
$$area\,\triangle AXC : area\,\triangle ABY=\frac{1}{2}\left(2a\right)\cdot7:\frac{1}{2}\left(3a\right)\cdot7=2:3$$

30. C Consider cases where $x+2=0$ or $\left|x^2-x-1\right|=1$. When $x=-2,\left(x^2-x-1\right)^{x+2}=1$. If $x^2-x-1=1$, then $x^2-x-2=0\Rightarrow x=2$ or $x=-1$. If $x^2-x-1=-1$, then $x+2$ must be even and $x^2-x-1=-1\Rightarrow x^2-x=0\Rightarrow x=0$ or $x=1$. If $x=0$, then $x+2$ is even. If $x=1$, then $x+2$ is odd and $(-1)^3\neq1$. The solutions are $x=-2, x=2, x=-1, x=0$.

31. A Using a graphing calculator, we find that the exponential regression gives $r=-0.988$, a stronger fit than any other line of best fit.

32. D Substituting -3 for x gives $2(-3)^2+(a-4)(-3)-2a=0\Rightarrow 30-5a=0\Rightarrow a=6$.

33. D In the diagram, let M be the midpoint of $\overline{AB}$.
Points P and Q trisect $\overline{AB}$ as shown.
(Insert Figure)
Let $AB=6x$. Then $AM=3x$. Since $AP=2x, PM=x$. But $OM=AM$, so $OM=3x$. Since $OP^2=PM^2+OM^2, 10^2=x^2+(3x)^2$ and



$x = \sqrt{10}$. Then the area of $\triangle AOB$ is $\frac{1}{2}(AB)(OM) = \frac{1}{2}\left(6\sqrt{10}\right)\left(3\sqrt{10}\right) = 90$.

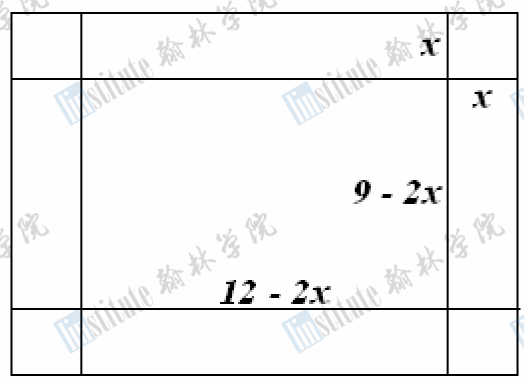
34. E. $2f(x) = 2f\left(2 \cdot \frac{x}{2}\right) = 2\left(\frac{2}{2+\frac{x}{3}}\right) = 2\left(\frac{4}{4+x}\right) = \frac{8}{4+x}$.

35. D Since m and n must both be positive, it follows that $n > 2$ and $m > 4$. Because $\frac{4}{m} + \frac{2}{n} = 1$ is equivalent to $(m-4)(n-2) = 8$, we need only find all ways of writing 8 as a product of positive integers. The four ways $1\cdot 8, 2\cdot 4, 4\cdot 2,$ and $8\cdot 1$, correspond to the four solutions $(m,n) = (5,10), (6,6), (8,4),$ and $(12,3)$.

36. B For real, unequal roots $b^2 - 4ac > 0$. So $k^2 - 4 > 0 \Rightarrow k = 3,4,5,6$. So the probability that $x^2 + kx + 1 = 0$ will have real, unequal roots is $\frac{4}{6}$ or $\frac{2}{3}$.

37. A Note that $C = A\log_{200} 5 + B\log_{200} 2 = \log_{200} 5^A + \log_{200} 2^B = \log_{200}\left(5^A \cdot 2^B\right)$, so $200^C = 5^A \cdot 2^B$. Therefore, $5^A \cdot 2^B = 200^C = \left(5^2 \cdot 2^3\right)^C = 5^{2C} \cdot 2^{3C}$. By uniqueness of prime factorization, $A = 2C$ and $B = 3C$. Letting $C = 1$, we get $A = 2, B = 3,$ and $A + B + C = 6$. The triplet $(A,B,C) = (2,3,1)$ is the only solution with no common factor greater than 1.

38. A Consider the figure as shown. The open box has length $12 - 2x$, width $9 - 2x$ and height x. So the volume $V = x(12-2x)(9-2x)$ is maximized when $x \approx 1.697$ and $V \approx 81.872$. This approximate solution was found using a graphing calculator. Once you have had some Calculus, you will be able to find the exact solution.



39. D Let x be the number of trees per acre and $y(x)$ the yield. Then

$$y(x) = \begin{cases} 600x, \text{ if } x \le 20 \\ \left[600 - 15(x-20)\right], \text{ if } x > 20 \end{cases}.$$

The maximum yield occurs when $x = 30, y = 135{,}000$.

40. A One must compute each probability:

a. $p(\text{even \& odd}) = \frac{3\cdot 3 + 3\cdot 3}{36} = \frac{1}{2}$

b. The success cases are (2,1), (2,3), (2,5), (3,4), (3,6), (5,4), (5,6), and (1,2), (3,2), (5,2), (4,3), (6,3) (4,5), (6,5). So $p = \frac{14}{36} = \frac{7}{18}$.

c. The success pairs are (1,5), (3,5), (5,5), (5,1), and (5,3) so $p = \frac{5}{36}$.

d. Since only (1,1), (3,3) and (5,5) are success cases, $p = \frac{3}{36} = \frac{1}{12}$.

e. $p = 0$.