Algebra I Solutions
State Mathematics Contest Finals, May 3, 2007

1. E. $3x+5=5(x+3) \Leftrightarrow 3x+5=5x+15 \Leftrightarrow -10=2x \Leftrightarrow x=-5$.

2. B. Every 7 days the day of the week repeats. $1000=7\cdot 142+6$, so we are one day short of 143 weeks, making the day of the week Wednesday.

3. A. $N \to 2N \to 2N+12 \to \dfrac{2N+12}{2}=N+6 \to (N+6)-N=6$.

4. D. A table of values that satisfy the second condition will give help us identify the numbers that satisfy this condition.

7 Mod 13	7	20	33	46	59	72	85	98	111	124
Mod 11	7	9	0	2	4	6	7	10	1	3

So we see that 59 has remainder 7 when divided by 13 but 4 when divided by 11. The sum of the digits is 14.

5. D. The slope of the segment joining the first two points is $\dfrac{13-5}{4-1}=\dfrac{8}{3}$, so the slope of the segment joining the next two points
$\dfrac{c-13}{9-4}=\dfrac{c-13}{5}=\dfrac{8}{3} \Rightarrow 3(c-13)=40 \Rightarrow 3c=79 \Rightarrow c=26\tfrac{1}{3}$

6. C. $15(t-1)=10(t+1) \Rightarrow 5t=25 \Rightarrow t=5$, so the time it should take him to get there is 5 hours, making the distance 60 km.

7. C. $\dfrac{7-\sqrt{5}}{3+\sqrt{5}}=\dfrac{7-\sqrt{5}}{3+\sqrt{5}}\cdot\dfrac{3-\sqrt{5}}{3-\sqrt{5}}=\dfrac{26-10\sqrt{5}}{4}=\dfrac{13}{2}-\dfrac{5}{2}\sqrt{5}$, so $ab=\dfrac{13}{2}\cdot-\dfrac{5}{2}=-\dfrac{65}{4}$.

8. B. We know that the largest *O* cab be is 1, since you cannot carry over more than 1. That makes $T=2$. So $\begin{array}{r} TO \\ \underline{+GO} \\ OUT \end{array} = \begin{array}{r} 21 \\ \underline{G1} \\ 1U2 \end{array}$, but now we see that *G = 8* in order to get the sum to carry 1, and can't be 9 as that would make *U = 1*. So we have $GOT-TUG=812-208=604$.

9. D. We want $4-x^2=c-x \Leftrightarrow x^2-x+c-4=0$. For this to have only one solution, the discriminant in the quadratic formula must be zero. So
$(-1)^2-4(1)(c-4)=0 \Leftrightarrow -4c=-17 \Rightarrow c=\dfrac{17}{4}=4.25$.

10. A. Don't try to find either x or y. Instead go directly for the product xy. Do this by squaring $(x+y)^2 = 4^2 \Leftrightarrow x^2 + 2xy + y^2 = 16$. Now subtract $x^2 + y^2 = 17$ to get $2xy = -1 \Rightarrow xy = -\frac{1}{2}$.

11. D. First notice that there is 30% left for the final exam. So we have $0.20(78) + 0.20(98) + 0.30(70) + 0.30F = 85$. Simplify this to get $15.6 + 19.6 + 21 + 0.3F = 85 \Rightarrow 0.3F = 28.8 \Rightarrow F = 96$

12. E.

$$y = \frac{2n+1}{n-1} \Rightarrow y(n-1) = 2n+1 \Leftrightarrow yn - y = 2n+1 \Leftrightarrow yn - 2n = 1 + y \Leftrightarrow$$
$$n(y-2) = 1 + y \Rightarrow n = \frac{1+y}{y-2} = \frac{y+1}{y-2}.$$

13. A. Look at a table of values of the powers of 3 and find the remainder when divided by 7.

Power	$3^0 = 1$	$3^1 = 3$	$3^2 = 9$	$3^3 = 27$	$3^4 = 81$	$3^5 = 243$	$3^6 = 729$	$3^7 = 2187$
Remainder	1	3	2	6	4	5	1	3

The pattern repeats every 6 powers, so $3^{2007} = 3^{334 \cdot 6 + 3}$, so the remainder is the same as 3^3, which is 6.

14. E. In 28 days it is covered, so on the 27$^{\text{th}}$ day it is half covered, on the 26$^{\text{th}}$ it is one-fourth covered, and on the 25$^{\text{th}}$ day is it one-eighth covered.

15. D. First, look at the tips: $0.15B + 0.20C + 1.00 = 6.30$. Now look at the meals: $B = C + 5, D = B + C$. Combining these we have $0.15(C+5) + 0.20C + 1.00 = 6.30 \Leftrightarrow 0.35C = 5.30 - 0.75 = 4.55 \Rightarrow C = 13$. This means that Barry's meal was 18 and Donald's was \$31.00.

16. C. Unlike Problem 10 where we went directly for the product xy, here we will need to find x and y individually and then find the sum. If we add the first equation to twice the second we get $x + 2y = 12, 2x - 2y = 6 \Rightarrow 3x = 18 \Rightarrow x = 6$. Now we find that $y = 3$, so the sum is 9.

17. A. First $a + b + c = 10$. Now subtract this from $3a + 2b + c = 17$ to get $2a + b = 7$. For this there are only two natural number solutions: $(a,b) = (1,5)$ or $(a,b) = (2,3)$. The first option yields $c = 4$, but this is smaller than b, so it does not work. The second option yields $c = 5$, which is okay. Now $a^2 + b^2 + c^2 = 2^2 + 3^2 + 5^2 = 38$

18. A. Let $F = C = \frac{9}{5}C + 32 \Rightarrow -\frac{4}{5}C = 32 \Rightarrow C = -40$.

19. B. $|3x+1| - 3 = x + 12 \Leftrightarrow |3x+1| = x + 15$, and we know that an absolute value expression can be either the positive or negative, so either $3x+1 = x+15$ or $3x+1 = -(x+15)$. Solving the first gives $3x+1 = x+15 \Leftrightarrow 2x = 14 \Rightarrow x = 7$. Solving the second we see that $3x+1 = -(x+15) \Rightarrow 4x = -16 \Rightarrow x = -4$, so the sum of the solutions is 3.

20. D. Since $f(x) = \frac{2x+1}{x-2}$, we know that

$$f(f(a)) = f\left(\frac{2a+1}{a-2}\right) = \frac{2\left(\frac{2a+1}{a-2}\right)+1}{\frac{2a+1}{a-2}-2} = \frac{\frac{4a+2}{a-2}+\frac{a-2}{a-2}}{\frac{2a+1}{a-2}-\frac{2(a-2)}{a-2}} = \frac{5a}{5} = a$$, but we are told that this equals 1, so $a = 1$.

21. E. Again the quickest way to answer this is to look at a table of values.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
n^2+1	2	5	10	17	26	37	50	65	82	101	122	145	170	197

We see that 13 will divide several of these numbers, namely 26 and 65, so the last statement is false. This does not prove that the others are always true, but since there is only one correct choice, we have found it. The proofs that the others are true is beyond most Algebra I students.

22. D. If $(x-y)^2 = x^2 - xy - y^2$, then

$x^2 - 2xy + y^2 = x^2 - xy - y^2 \Rightarrow 2y^2 = xy \Rightarrow 2y^2 - xy = 0$, so

$y(2y-x) = 0 \Rightarrow y = 0 \Rightarrow \frac{y}{x} = 0 \ \ or\ 2y - x = 0 \Rightarrow \frac{y}{x} = \frac{1}{2}$

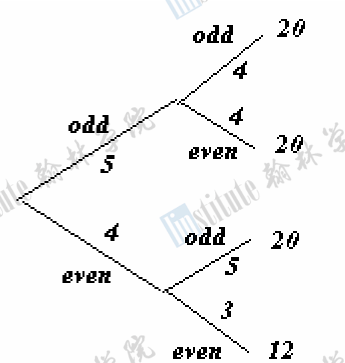
23. A. The slope of the line is $m = \frac{51-36}{0-5} = \frac{15}{-5} = -3$ and the y-intercept is given as 51, so one equation for the line is $y = -3x + 51$. The x-intercept occurs when $y = 0$, so we have $0 = -3x + 51 \Rightarrow 3x = 51 \Rightarrow x = 17$.

24. B. The value of the number abc is really $100a + 10b + c$, so
$abc - cba = (100a + 10b + c) - (100c + 10b + a) = 99a - 99c = 9 \cdot 11(a-c)$.
Clearly from this both 9 and 11 divide the number. Also $a - c < 8$, since

$a > c > 0$, so the product must be $9 \cdot 11(a-c) \le 9 \cdot 11 \cdot 8 = 792$. If we check all possible products we see that the ten's digit is always 9. This number could actually be 99 if $a - c = 1$. Thus B is the only choice which is not necessarily true.

25. C. This is another one of those problems where you might want to go directly to the desired sum instead of finding the individual values for *x*, *y*, and *z*. Notice that when the two equations are added we get $5x + 5y + 5z = 15 \Rightarrow x + y + z = 3$.

26. D. Let's draw a simple "tree diagram" for this. We see that there are a total of 72 possibilities, but that only 40 result in an odd sum. Thus the probability is $\frac{40}{72} = \frac{5}{9}$.



27. C. Since $C + E = 117$ and $C - E = 3$, we see that $2C = 120 \Rightarrow C = 60$ and $E = 57$.

28. E. The number of coins is 14 so $Q + D + N = 14$. We are told that the value of the quarters (25Q) is equal to the value of the other coins, so $25Q = 10D + 5N \Leftrightarrow 5Q = 2D + N \Rightarrow N = 5Q - 2D$. So we can combine this with the first equation to get $Q + D + (5Q - 2D) = 14 \Leftrightarrow 6Q - D = 14$. There are infinitely many solutions to this last equation $(Q, D) = (3,4), (4,10), (5,16), \cdots$, but only the first will result in 14 coins allowing for at least one nickel. Thus the number of nickels is $N = 5Q - 2D = 5(3) - 2(4) = 7$.

29. C. Since $2^x = 15$, $(2^x)^3 = 2^{3x} = 8^x = 15^3$. So now $\sqrt{0.6 \cdot 8^x} = \sqrt{\frac{6 \cdot 15^3}{10}} = \sqrt{\frac{3 \cdot 15^3}{5}} = \sqrt{3 \cdot 3 \cdot 15^2} = 45$.

30. B. Let *T* be the total he started with. He first spends one-fifth of this so he has $\frac{4}{5}T$ remaining. He again spends one-fifth of what he has, so the has four-fifths of this amount, or $\left(\frac{4}{5}T\right)\frac{4}{5} = \frac{16}{25}T$ left. We are told that he has spent \$36, so we have $T - \frac{16}{25}T = 36 \Leftrightarrow \frac{9}{25}T = 36 \Rightarrow 9T = (36)(25) \Rightarrow T = 100$

31. A. You are allowed to repeat letters and numbers so the total is $26^3 \cdot 10^4 \approx 1.76 \times 10^8$

32. A. Since $f(x)=2x+1$, $f(3x+7)=2(3x+7)+1=6x+15$. If this equals 13, we have $6x+15=13 \Rightarrow 6x=-2 \Rightarrow x=-\frac{1}{3}$.

33. D. The point (0,1) must satisfy both equations, which is does. Substituting $mx+1$ in for *y* in the equation $x^2+y^2=1$ yields $x^2+(mx+1)^2=1$. Solve this for *x* to get both solutions. $x^2+m^2x^2+2mx+1=1 \Rightarrow x^2(1+m^2)+2mx=0$. Now factor the last expression to get $x\left[x(1+m^2)+2m\right]=0$ and we see that either $x=0$ (the solution we already have), or $x(1+m^2)+2m=0 \Rightarrow x=-\frac{2m}{1+m^2}$.

34. C. Let *x* be the amounts that Chris and Pat put into their accounts initially. For Chris we have $\left(x-\frac{1}{2}x\right)1.06+200$. For Pat we have $(x+200)1.06-\frac{1}{2}x$. We are told that at this point Pat has 300 more that Chris, so $(x+200)1.06-\frac{1}{2}x-300=\left(x-\frac{1}{2}x\right)1.06+200$. This simplies to $1.06x+212-.5x-300=0.53x+200 \Rightarrow 0.56x-88=0.53x+200 \Rightarrow 0.03x=288$. Finally we solve for *x* and get $x=288\div 0.03=9600$.

35. C. Let *r* and 2*r* be the rates that the two candles burn. We know $r=\frac{1}{6}$ of the candle per hour and $2r=2\left(\frac{1}{6}\right)=\frac{1}{3}$. We want the time *t* for which $1-\frac{1}{6}t=3\left(1-\frac{1}{3}t\right)$. So $1-\frac{1}{6}t=3\left(1-\frac{1}{3}t\right) \Leftrightarrow 1-\frac{1}{6}t=3-t \Leftrightarrow t-\frac{1}{6}t=3-1 \Leftrightarrow \frac{5}{6}t=2 \Rightarrow t=\frac{12}{5}hrs$, but this would be 2 hours and 24 minutes.

36. B. Since $2^{1000}=2^{3\cdot 333\frac{1}{3}}=8^{333\frac{1}{3}}$ as well as $2^{1000}=2^{4\cdot 250}=16^{250}$, so $8^{333}<2^{1000}<16^{250}$. But we want $8^{333}<10^{???}<16^{250}$. Notice that as the base increases, the exponent decreases, to the exponent we want must be between 333 and 250. The only one listed is 301. (When you learn logarithms in Algebra II, there is a better way to solve this problem.)

37. D. The complete expansion this binomial is $(x+2a)^5=x^5+5\cdot x^4\cdot 2a+10\cdot x^3\cdot(2a)^2+10\cdot x^2\cdot(2a)^3+5\cdot x\cdot(2a)^4+(2a)^5$, but we only need the third terms which is $10\cdot x^3\cdot(2a)^2=40x^3a^2$.

38. E. If n is even, the middle two numbers will be $n^2 \pm 1$. Starting at n^2+1 go back by twos to find the odd numbers before this. How many twos, well just $\frac{n}{2}$, so we get $n^2+1-\frac{n}{2}\cdot 2 = n^2-n+1$. If n is odd, the middle number will always be $\frac{n^3}{n}=n^2$. From this we need to subtract $\frac{n-1}{2}$ twos, yielding $n^2-\left(\frac{n-1}{2}\right)2 = n^2-n+1$.

39. C. First we have to assume that a blank is incorrect. Of course on some multiple choice tests you might get points for blanks to discourage guessing, but this must not be multiple choice. So what we have is $5C-3I=16$ and $C+I=40$. Solving this system by adding $5C-3I=16$ to $3C+3I=120$ to yield $8C=136 \Rightarrow C=17 \Rightarrow I=23$.

40. E. The equation as stated is $x-\frac{1}{x}=1$, so each of the correct choices must have a solution which is also a solution to this equation. To solve the original equation, multiply by x yielding $x^2-1=x \Leftrightarrow x^2-x-1=0 \Rightarrow x=\frac{1\pm\sqrt{5}}{2}$. Choices (a), (b), and (c) all simplify to this same equation. Choice (e), $x=\frac{1-x}{x} \Leftrightarrow x^2+x-1=0 \Rightarrow x=\frac{-1\pm\sqrt{5}}{2}$, which are different, so (e) does not work. What about choice (d)? If $x^3=2x+1 \Leftrightarrow x^3-2x-1$, and cubes are a lot harder to solve, unless we can find one root by graphing or inspection. Fortunately for us, -1 is a solution, so $(x+1)$ will be a factor of x^3-2x-1. Long division would give the answer, but it better be the quadratic that we had above. Checking we see that $(x+1)(x^2-x-1)=x^3+x^2-x^2-x-x-1=x^3-2x-1$.