# NC STATE MATHEMATICS CONTEST – APRIL 2006: SOLUTIONS

1. Answer **a**): $2^{\log_4(8)} = 16^x \Rightarrow \log_4(8) = x\log_2(16) \Rightarrow x = (3/2)/4 = 3/8$.
2. Answer **b**): Graph and trace the functions $x$ and $\sin(x)$.
3. Answer **c**): The pentagon can be viewed as a trapezoid with bases of length 3 and 4 and height 3 with a triangle of base 3 and height 2 removed. The area is (7/2)(3) – 3 = 15/2.
4. Answer **d**): $1 - \frac{9(9!)}{9(10^9)} = 0.99963712$.
5. Answer **e**): If $s$ is the length of a side of the larger square and a vertex of the smaller square divides this side into segments of length $x$ and $s – x$, the area of the smaller square is $x^2 + (s-x)^2$ which is minimized when $x = s/2$ and then the area of the smaller square is half the area of the larger square.
6. Answer **c**): The Law of Sines gives $\frac{4}{|AD|} = \frac{3}{|BD|}$. Since $6 = |AD| + |BD| = \left(1 + \frac{3}{4}\right)|AD|$, we see that $|AD| = \frac{24}{7}$.
7. Answer **a**): With $z = x + iy$, we have $\frac{1}{1-z} = \frac{1-x+iy}{2-2x} = \frac{1}{2} + i\frac{y}{2-2x}$.
8. Answer **e**): $x + \frac{1}{x} = n\pi$ has a solution in the interval $(0,\pi)$ for infinitely many $n$.
9. Answer **c**): $\frac{\pi(4^2 - 3^2)}{\pi 5^2} = \frac{7}{25}$.
10. Answer **c**): $4 = (x-10)^2 + (mx-5)^2 = (1+m^2)x^2 - (20+10m)x + 125$ has exactly two solutions if $(20+10m)^2 - 4(1+m^2)(121) = 0$ which is solved by 7/24 and 3/4. We want the larger of these two.
11. Answer **b**): We use Heron's Formula with semiperimeter = 16. We obtain $\sqrt{16(16-6)(16-12)(16-14)} = 16\sqrt{5}$.
12. Answer **d**): Every twelve consecutive terms sum to 0 and 2006 = 12(267) + 2 so we get $\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{2\pi}{6}\right) = \frac{1+\sqrt{3}}{2}$.
13. Answer **a**): The quadratic defines a sphere of radius 8 and center (5,6,4). The linear equation defines a plane at distance greater than 8 from the center of the sphere.
14. Answer **b**): Since $(n-1) + n + (n+1) = 3n$, (I) is true. Since $(n-1)^2 + n^2 + (n+1)^2 = 3n^2 + 2$, (II) is false. Since $(n-1)^3 + n^3 + (n+1)^3 = 3n^3 + 6n = 3n(n^2+2)$, (III) is false.
15. Answer **d**): Two distinct diagonals meet at the center of the cube and, with an edge of the cube, form an isosceles triangle whose sides are $s$, $s\sqrt{3}/2$ and $s\sqrt{3}/2$. The degree measure of the angle between the two equal sides is 70.53.

OVER FOR SOLUTIONS TO THE REMAINING PROBLEMS.

16. Answer **c**): Since $x_{2n+1} = x_0/2,\ x_{2n+2} = (x_0/2)(n+3)/(n+2)$ so $x_{22} = 72(10+3)/(10+2) = 78$.

17. Answer **a**): Cut the vertical seams of the box and lay the four vertical faces flat obtaining a 2 by 3 rectangle with two 2 by 1 rectangles and two 3 by 1 rectangles appended to its edges. It is then clear that the fly has a path of length 5.

18. Answer **c**): If $CC'$ is a diameter of the circle and $A$ and $B$ are in the same arc from $C$ to $C'$, the three points are in a semicircle (probability 1/2). If $A$ and $B$ are not in the same arc from $C$ to $C'$, $A$ must be in the same arc from $B$ to $B'$ as $C$ (probability (1/2)(1/2) = 1/4.)

19. Answer **e**): With $u^2 = (x-1)/27,\ x \pm (x+8)\sqrt{(x-1)/27} = 1 + 27u^2 \pm (9 + 27u^2)u = (1 \pm 3u)^3$. Our expression equals $6u$. Since $u = \frac{1}{3}\sqrt{\frac{x-1}{3}}$, $u$ is rational when $\frac{x-1}{3}$ is a perfect square and $x < 2007$. There are 25 such $x$'s.

20. Answer **d**): Let $\alpha$ (respectively $\beta$) be the central angle associated with the side of length 10 (20). Then, from the Law of Cosines, $50 = r^2(1-\cos\alpha)$ and $200 = r^2(1-\cos\beta)$. Also, $4\alpha + 2\beta = 2\pi$. Solving for $r$, we find 13.6603.

## Integer Answer Problems

1. Answer: **997** The number of steps is one less than the number of piles and 998 is the largest possible number of piles, 997 with 1 coin and 1 with two coins.
2. Answer: **2** Paint opposite faces of a single cube black.
3. Answer: **150** Let $d$ be the distance. Then $d/20$ covered in 10 min. at speed 15/20 miles per min.
4. Answer: **152** We need $9(9/10)^n\left(1-(9/10)^n\right) < 10^{-6}$; 152 is the smallest integer that does the job.
5. Answer: **1** There are exactly two points that satisfy the two equations and the slope of the line joining them is 1.
6. Answer: **4** The equation can be written $(x^2 - a)^4 = 1$ so $x^2 = a \pm 1$, and we get 4 real solutions.
7. Answer: **9** Algebraic manipulation shows that the sum of the first entry in the second row and the last entry in the third row must be 19. Therefore, the first entry in the second row must be at least 7. However, since the second row cannot sum to 26 with 7 as a summand, the next smallest integer available for the first entry in the second row is 9. A solution does exist with 9 in this position.
8. Answer: **63,504** Our possible "moves" are NE, NW, SW and SE (directions). We need equal numbers of NE and SW moves and equal numbers of NW and SE moves and identify a sequence with a list of 10 moves in any order. We get $\sum_{k=0}^{5}\binom{10}{k}\binom{10-k}{k}\binom{10-2k}{5-k}\binom{5-k}{5-k} = \binom{10}{5}^2 = 63{,}504$
9. Answer: **36** Since $7! = 2^4 3^2 5^1 7^1$, we have 8 primes to distribute to $a$, $b$, $c$ and $d$, and each receive two distinct primes. Thus, $\{a,b,c,d\} = \{2\cdot 3,\ 2\cdot 3,\ 2\cdot 5,\ 2\cdot 7\}$.
10. Answer: **2** Since $41160 = p(13) = a_0 + a_1 13 + \cdots a_n 13^n$ where the $a$'s are the coefficients of $p(x)$ and 41160 in base 13 is $(15972)_{13}$, $p(x) = x^4 + 5x^3 + 9x^2 + 7x + 2 = (1+x)^3(2+x)$.

TIE BREAKER ANSWER: There are 9 different ways to fill in the grid.