NC STATE MATHEMATICS CONTEST – APRIL 2006

PART I: 20 MULTIPLE CHOICE PROBLEMS

1. Find the value of x that satisfies the equation $2^{\log_4(8)} = 16^x$.

a) 3/8 b) 1/2 c) $2^{3/32}$ d) $2^{1/8}$ e) None of a) through d) is correct.

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2. Consider the set of all positive solutions to the equation $x = 100\sin(x)$ where x is measured in radians. Which of the following is correct?

a) The largest number in the set is 95.5 to the nearest tenth.

b) The largest number in the set is 96.1 to the nearest tenth.

c) The largest number in the set is 98.5 to the nearest tenth.

d) The largest number in the set is 98.8 to the nearest tenth.

e) The largest number in the set is 99.1 to the nearest tenth.

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3. A pentagon *ABCDE* has vertices *A*(0,0), *B*(4,0), *C*(2,2), *D*(4,3) and *E*(1,3), Which of the following is closest to the area of the pentagon?

a) 6.5 b) 7.0 c) 7.5 d) 8.0 e) 8.5

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4. Suppose a 10 digit positive integer (leading digit not 0) is chosen at random. What is the probability that the digits in this integer are NOT distinct?

a) 99.9937% b) 99.9837% c) 99.9737% d) 99.9637% e) 99.9537%

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5. A square is inscribed in a larger square. (That is, the four vertices of the inscribed square lie on the four sides of the larger square.) What is the smallest possible value of the ratio of the area of the inscribed square to that of the larger square?

a) 1/4 b) 1/3 c) $\frac{1}{\sqrt{2}}$ d) $\frac{2}{\sqrt{5}}$ e) None of a) through d) is correct.

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6. ΔABC has sides *AB* of length 6, *BC* of length 3 and *AC* of length 4. The angle bisector of $\angle C$ intersects side *AB* in point *D*. What is the length of the segment *AD*?

a) 18/7 b) 3 c) 24/7 d) 7/2 e) None of a) through d) is correct.

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7. Let z denote a complex number and define $S = \left\{ \frac{1}{1-z} : |z| = 1 \text{ and } z \neq 1 \right\}$. Which of the following best describes the set S, when S is interpreted geometrically as a set of points in the complex plane?

a) S is a straight line parallel to the imaginary axis.

b) S is a straight line parallel to the real axis.

c) S is a circle with a single point missing.

d) S is a parabola.

e) S is a branch of a hyperbola.

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8. Let X be the set of all solutions to the equation $\cos(x)\sin\left(x + \frac{1}{x}\right) = 0$ with $0 < x < \pi$. How many real numbers does the set X contain?

a) 0 b) exactly 1 c) exactly 2 d) more than 2 but finitely many

e) None of a) through d) is correct.

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9. Five concentric circles with radii 1, 2, 3, 4 and 5 are drawn on a flat surface. The circles with radii 1 through 4 divide the circle of radius 5 into five regions: a circle of radius 1 and 4 annuli (rings). The first annulus has inner radius 1 and outer radius 2, the second has inner radius 2 and outer radius 3, and so on. A point is chosen at random in the circle of radius 5. What is the probability that the point lies in the annulus whose inner radius is 3 and whose outer radius is 4?

a) 3/25 b) 1/5 c) 7/25 d) 9/25 e) None of a) through d) is correct.

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10. What is the largest value of m such that the graph of the equation $y = mx$ meets the graph of the equation $(x-10)^2 + (y-5)^2 = 4$?

a) 7/24 b) 1/2 c) 3/4 d) There is no m for which the graphs meet.

e) None of a) through d) is correct.

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11. A car is traveling on flat ground. It leaves Point A and travels 6 miles in a straight line to Point B. It then turns and travels 12 miles in a straight line to Point C. Finally, the car turns again and travels 14 miles in a straight line back to Point A. What is the area of the triangle whose vertices are A, B and C?

a) $8\sqrt{5}$ b) $16\sqrt{5}$ c) $24\sqrt{130}$ d) $48\sqrt{130}$

e) The answer cannot be determined uniquely from the given information.

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12. What is the value of the sum $\cos\left(\frac{\pi}{6}\right)+\cos\left(\frac{2\pi}{6}\right)+\cos\left(\frac{3\pi}{6}\right)+\cdots+\cos\left(\frac{2006\pi}{6}\right)$, using radian measure?

a) 0 b) 1003 c) $1003\sqrt{3}$ d) $\frac{1+\sqrt{3}}{2}$ e) None of a) through d) is correct.

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13. Which of the following most accurately describes the intersection of the two surfaces

$$x^2+y^2+z^2-10x-12y-8z=-13 \text{ and } x+y+z=1?$$

a) There are no points of intersection.
b) The intersection is a single point.
c) The intersection is a line.
d) The intersection is a parabola.
e) The intersection is a circle.

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14. Let $N=3\left(2^{2006}\right)$. Consider the following three statements.

(I) There exist three consecutive integers whose sum is *N*.
(II) There exist three consecutive integers the sum of whose squares is *N*.
(III) There exist three consecutive integers the sum of whose cubes is *N*.

Which of (I), (II) and (III) are true?

a) None are true
b) Only (I) is true
c) Only (I) and (II) are true.
d) Only (I) and (III) are true.
e) All three are true.

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15. A *diagonal* of a cube is a line segment of maximum length whose endpoints are vertices of the cube. To the nearest degree, what is the angle of intersection θ, $0<\theta\le 90^\circ$ between two distinct diagonals of the same cube?

a) 45° b) 53° c) 67° d) 71° e) 90°

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16. The terms in a sequence x_n alternately increase and decrease in accordance with the equations

$$x_0=144,\ x_{2n+1}=x_{2n}\left(1-\frac{1}{n+2}\right),\ x_{2n+2}=x_{2n+1}\left(1+\frac{1}{n+2}\right) \text{ for } n=0,1,\ldots$$

What is the value of x_{22} to the nearest integer?

a) 144 b) 80 c) 78 d) 72 e) None of a) through d) is correct.

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17. A rectangular box with no top has base a 2 ft by 3 ft rectangle and volume 6 cubic ft. A fly crawls from one corner at the top of the box to the diagonally opposite corner at the top of the box. What can be said about the minimum possible distance D the fly can crawl, provided **THE FLY'S PATH TAKES IT INTO THE BASE OF THE BOX**?

a) $0 < D \le 5$
b) $5 < D \le \sqrt{5} + \sqrt{10}$
c) $\sqrt{5} + \sqrt{10} < D \le 2 + \sqrt{13}$
d) $2 + \sqrt{13} < D \le \sqrt{5} + 4$
e) $\sqrt{5} + 4 < D \le 7$

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18. Points *A, B* and *C* are independently and randomly placed on the boundary of a circle. What is the probability that the three points will lie in some semicircle of the circle?

a) 3/8 b) 1/2 c) 3/4 d) 7/8 e) None of a) through d) is correct.

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19. Determine the number of integers $1 \le x \le 2006$ such that the expression

$$\sqrt[3]{x+(x+8)\sqrt{\frac{x-1}{27}}}-\sqrt[3]{x-(x+8)\sqrt{\frac{x-1}{27}}}$$

is a rational number.

a) 5 b) 10 c) 15 d) 20 e) 25

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20. A hexagon is inscribed in a circle of radius *r*. Suppose that four of the edges of the hexagon are ten feet long and two of the edges are twenty feet long, but the exact arrangement of the edges is unknown. What is the value of *r* to three decimal places?

a) 10.673 feet b) 11.537 feet c) 12.664 feet d) 13.660 feet
e) The answer cannot be determined uniquely from the given information.

THIS CONCLUDES PART I. PART II BEGINS ON THE FOLLOWING PAGE.

PART II: 10 INTEGER ANSWER PROBLEMS

1. Starting with a single pile of 999 coins, a person does the following in a series of steps: On step one, he splits the pile into two nonempty piles. Thereafter, at each step, he chooses a pile with 3 or more coins and splits this pile into two piles. What is the largest number of steps that is possible?

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2. The 48 faces of 8 unit cubes are painted white. What is the smallest number of these faces that can be repainted black so that it becomes impossible to arrange the 8 unit cubes into a two by two by two cube, each of whose 6 faces is totally white?

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3. A parent and child are on a trip from Point A to Point B. The parent drives at a constant speed on the highway from A to B. Every 10 minutes during the trip the child asks "Are we there yet?" At one point during the trip, the parent answers, "We are 60% of the way there." while noticing that the car is adjacent to mile marker 240. The next time the child asks the question, the parent answers, "We are 65% of the way there." The next time the child asks the question the parent does not answer but does notice that the car is adjacent to mile marker 255. What is the distance in miles between Point A and Point B?

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4. What is the smallest positive integer n such that $\left(\frac{9}{10}\right)^{n+1}+\left(\frac{9}{10}\right)^{n+2}+\cdots+\left(\frac{9}{10}\right)^{2n}<\frac{1}{10^6}$?

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5. Let S denote the set of points of intersection of the hyperbola $xy=2$ and the graph of $y=\sqrt[3]{x^3-20}$. How many lines of slope 1 pass through at least one point of S?

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6. How many non-real solutions are there to the equation $x^8-4ax^6+6a^2x^4-4a^3x^2+a^4=1$ if a is a real number and $a>1$?

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7. A 4 by 4 grid has a zero in each of its four corners. Suppose that the remaining 12 positions are filled with the integers 1, 2, … , 12 with each integer used exactly once so that both of the following conditions are satisfied:

   (i) The second entry in the first row is 8, the last entry in the second row is 1, the first entry in the third row is 6, and the third entry in the last row is 5.

   (ii) Any row or column with four nonzero entries sums to 26.

   What is the smallest possible value that can appear as the first entry in the second row, i.e. the entry that is directly above the 6?

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8. A point $P = (a,b)$ in the plane is called a lattice point if both its coordinates are integers. Suppose $P_0 = (0,0)$, $P_1, P_2, \ldots, P_9, P_{10} = (0,0)$ is a sequence of lattice points such that the distance between P_i and P_{i+1} is $\sqrt{2}$ for $i = 0,1, \ldots ,9$. How many such sequences are there?

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9. Suppose that 7! is written as a product *abcd* where *a, b, c* and *d* are positive integers such that each of *a, b, c* and *d* has the same number of positive integral divisors. [As an example, 6 has four positive divisors 1, 2, 3 and 6.] What is the largest possible value of $a+b+c+d$?

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10. The coefficients of a polynomial *p(x)* are nonnegative, single-digit integers. If $p(13) = 41160$, then what is the smallest positive integer *n* such that the interval $[-n,0]$ contains all the roots of *p(x)*?

The following problem will be used only as part of a tie-breaking procedure. You should not work on it until you have completed the rest of the test.

TIE BREAKER PROBLEM

Consider the grid in Integer Answer problem #7 with a zero in each of its four corners. How many different ways are there to fill in the remaining twelve positions with the integers 1 through 12, with each digit used exactly once and with both conditions (i) and (ii) satisfied?