2005 State Mathematics Contest – Algebra II
Solutions

1. D. A line is a set of points for which the slope is constant, so
$$\frac{12-4}{6-1}=\frac{12-10}{6-c}\Leftrightarrow\frac{8}{5}=\frac{2}{6-c}\Rightarrow 8(6-c)=10\Rightarrow 6-c=\frac{5}{4}\Rightarrow c=6-\frac{5}{4}=4.75.$$

2. C. $f(7)=f(7-1)+7+1=f(6)+8=\left(f(6-1)+6+1\right)+8=f(5)+15=7+15=22$

3. B. $83(0.75)+x(0.25)=85\Rightarrow x(0.25)=85-62.25=22.75\Rightarrow x=91$

4. C. To solve this, we first must square both sides to remove the first radical, yielding $4x+\sqrt{17x^2+2}=\left(x+2\right)^2=x^2+4x+4\Rightarrow\sqrt{17x^2+2}=x^2+4$. No we need to square again to get rid of the remaining radical. So $17x^2+2=\left(x^2+4\right)^2\Leftrightarrow 17x^2+2=x^4+8x+16\Leftrightarrow x^4-9x^2+14=0$. We can solve this by factoring, so $x^4-9x^2+14=0\Leftrightarrow\left(x^2-2\right)\left(x^2-7\right)=0\Rightarrow x=\pm\sqrt{2},\pm\sqrt{7}$. But when you square both sides of an equation, you can introduce extraneous roots. Clearly $x=-\sqrt{7}$ cannot be a solution since the right side of the original equation would be negative, so we should check $-\sqrt{2}$ as well. Substituting this into the original equation yields $\sqrt{-4\sqrt{2}+\sqrt{17(-\sqrt{2})^2+2}}=\sqrt{-4\sqrt{2}+\sqrt{36}}=\sqrt{-4\sqrt{2}+6}=\sqrt{\left(-\sqrt{2}+2\right)^2}=-\sqrt{2}+2$, which is fine. But what about the other "solutions" Again trying each of these yields $\sqrt{4\sqrt{2}+\sqrt{17(2)+2}}=\sqrt{4\sqrt{2}+\sqrt{36}}=\sqrt{4\sqrt{2}+6}=\sqrt{\left(\sqrt{2}+2\right)^2}=\sqrt{2}+2$, so this one checks. Similarly, $\sqrt{4\sqrt{7}+\sqrt{17(7)+2}}=\sqrt{4\sqrt{7}+\sqrt{121}}=\sqrt{4\sqrt{7}+11}=\sqrt{\left(\sqrt{7}+2\right)^2}=\sqrt{7}+2$, so this one checks as well, so the sum of the solutions is $-\sqrt{2}+\sqrt{2}+\sqrt{7}=\sqrt{7}$.

5. A. If $\frac{x+a}{x+b}=c\Rightarrow x+a=c(x+b)=cx+bc\therefore cx-x=a-bc\Rightarrow x(c-1)=a-bc$, so $x=\frac{a-bc}{c-1}=\frac{cb-a}{1-c}$.

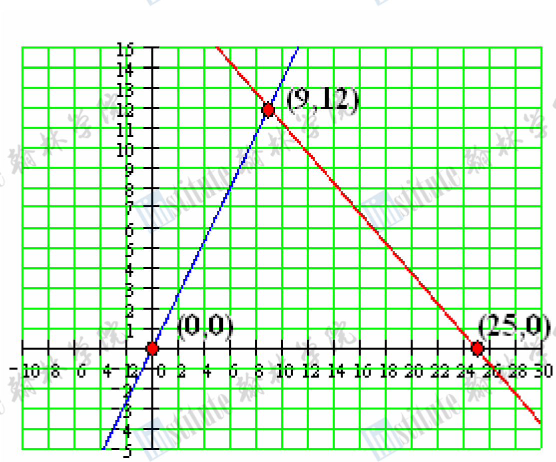
6. E. If $f^{-1}\left(\frac{1}{x+1}\right)=2x-3$, then $f(2x-3)=\frac{1}{x+1}$. Now let $u=2x-3$, so $x=\frac{u+3}{2}$ and $f(u)=\frac{1}{\frac{u+3}{2}+1}=\frac{1}{\frac{u+5}{2}}=\frac{2}{u+5}$, or $f(x)=\frac{2}{x+5}$.

7. B. This number is too big for most calculators, so we have to come up with some other method for solving it. If we look at powers of 7, they are 0, 7, 9, 3, 1, 0, … cycling through in groups of 5. We need to know the remainder when 7^7 is divided by 5, and it turns out to be a 3, so ones digit is the same as 7^5, which is 3.

8. E. $$\sqrt{\frac{x}{y}\sqrt{\frac{y^3}{x^3}\sqrt{\frac{x^5}{y^5}}}} = \sqrt{\frac{x}{y}\sqrt{\frac{y^3}{x^3}\cdot\frac{x^{5/2}}{y^{5/2}}}} = \sqrt{\frac{x}{y}\sqrt{\frac{y^{1/2}}{x^{1/2}}}} = \sqrt{\frac{x}{y}\cdot\frac{y^{1/4}}{x^{1/4}}} = \sqrt{\frac{x^{3/4}}{y^{3/4}}} = \frac{x^{3/8}}{y^{3/8}} = \left(\frac{x}{y}\right)^{3/8}$$

9. D. The best way to look at this is to get an expression which is less than zero, so $\frac{x^2+1}{x+2} < \frac{x+5}{2} \Leftrightarrow \frac{x^2+1}{x+2} - \frac{x+5}{2} < 0 \Leftrightarrow \frac{2(x^2+1)-(x+2)(x+5)}{2(x+2)} < 0$. Simplify the numerator to get $\frac{2x^2+2-(x^2+7x+10)}{2(x+2)} < 0 \Leftrightarrow \frac{x^2-7x-8}{2(x+2)} < 0 \Leftrightarrow \frac{(x-8)(x+1)}{2(x+2)} < 0$.
This last expression has three places where the sigh changes. They are 8, -1, and -2, and since each factor involving these values has power 1, the sign actually changes at each of these. If $x > 8$, all of the factors are positive, so the rational expression is also positive. As we move down the number line the sign changes to negative between 9 and -1, back to positive between -1 and -2, and back to negative when $x < -2$.

10. B. A quick sketch of the function shows the triangle in question. Two of the vertices are (0,0) and (25,0) and the third is the intersection of the two oblique lines. By solving the system of equations $3x+4y=75$ and $4x-3y=0$, we find the solution to be (9,12), which is the final vertex of the triangle. Now the area is $\frac{1}{2}bh = \frac{1}{2}(25)(12) = 150$.



11. A. If $x > 1/4$, $|4x-1| = 4x-1$, otherwise it equals $-(4x-1)$, so we need to find solutions for $x^2+x+1=4x-1$ and $x^2+x+1=-4x+1$. The solutions to the first equation, $x^2+x+1=4x-1 \Leftrightarrow x^2-3x+2=0 \Leftrightarrow (x-2)(x-1)=0$ are 2 and 1, and since both are greater than 1/4th, both are valid solutions. The solutions to the second equation $x^2+x+1=-4x+1 \Leftrightarrow x^2+5x=0 \Leftrightarrow x(x+5)=0$ are 0 and -5, and, again, since both are less than 1/4th, both are solutions. The average, then, of the four solutions is $\frac{1+2+0+(-5)}{4} = \frac{-2}{4} = -\frac{1}{2}$.

12. D. $X^2 \oplus (2X \oplus 1) = X^2 \oplus \left(\frac{2X}{2X+1}\right) = \frac{X^2\left(\frac{2X}{2X+1}\right)}{X^2 + \left(\frac{2X}{2X+1}\right)} = \frac{2X^3}{2X^3 + X^2 + 2X}$, so we need to solve $\frac{2X^3}{2X^3+2X^2+X} = \frac{1}{4} \Rightarrow 8X^3 = 2X^3 + 2X^2 + X \Leftrightarrow 6X^3 - X^2 - 2X = 0$. Factor this last expression to get $X\left(6X^2 - X - 2\right) = X\left(3X-2\right)\left(2X+1\right) = 0$, making $X = 0$, $2/3$ or $-1/2$. Since the operation is only defined for positive reals, 2/3 is the only correct answer.

13. C. Using the operation in 13, we can see the progression of values for $A \oplus A, A \oplus A \oplus A, A \oplus A \oplus A \oplus A, \cdots$. These values are $\frac{A^2}{2A} = \boxed{\frac{A}{2}}, \frac{\left(A/2\right)A}{\left(A/2\right)+A} = \frac{A^2}{3A} = \boxed{\frac{A}{3}}, \frac{\left(A/3\right)A}{\left(A/3\right)+A} = \frac{A^2}{4A} = \boxed{\frac{A}{4}}, \ldots$, so it looks like the general term would be $\frac{A}{n}$.

14. C. Since the square roots are infinite, we can rewrite this expression as $\sqrt{p+\sqrt{p+\sqrt{p+\sqrt{p+\cdots}}}} = 7$ as $S = \sqrt{p+S}$, where $S = \sqrt{p+\sqrt{p+\sqrt{p+\cdots}}}$, so $S = \sqrt{p+S} = 7 \Rightarrow 7^2 = p + 7 \Rightarrow p = 42$.

15. B. so the only real roots come from the first factor, and the product of these roots must be -8.

16. E. The 30 mile uphill section took 3 hours while the downhill 25 miles took h hours, while the average speed was $\frac{55}{3+h} = 15 \Rightarrow 55 = 45 + 15h \Rightarrow h = 2/3$ hours. The average speed downhill then was $\frac{25}{2/3} = \frac{75}{2} = 37.5$.

17. A. $f(g(x)) = f(x+a) = \left(x+a\right)^2 - 12(x+a) + 5 = x^2 + \left(2a-12\right)x + \left(a^2 - 12a + 5\right)$. Since we are told that $f\left(g(x)\right) = x^2 + c$, we know that $2a - 12 = 0$ and $a^2 - 12a + 5 = c$, so $a = 6$ and $a^2 - 12a + 5 = 36 - 72 + 5 = -31 = c$.

18. C. This is a recursive equation, with the first value being 0 and each successive value is $A_{n+1} = round(0.90A_n + 10), A_0 = 0$. The values at the end of each day will be as follows:

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
10	19	27	34	41	47	52

19. E. When a graph is flipped over the x-axis, $f(x)$ turns into $-f(x)$. To move a graph to the right, we must subtract from the x-value. To move a graph up we add to the final value, so the desired formula is $-f(x-3)+5=5-f(x-3)$.

20. D. Trying to go directly from the sequence to the formula is difficult, but since there are 5 choices, we can try each one. Clearly (a) fails as the third term is 12. (b) starts fine but fails on the 4th term, which is 14. (c) grows without limits, so it fails, and (d) works.

21. E. The remainders when power of 3 are divided by five are as follows: $3^0 \div 5 \to 1; 3^1 \div 5 \to 3; 3^2 \div 5 \to 4; 3^3 \div 5 \to 2; 3^4 \div 5 \to 1; 3^5 \div 5 \to 3; \cdots$, so you can see that these cycle through 1, 3, 4, 2, 1,3,4,2…. Since 98 has remainder 2 when divided by 4, the remainder will be the same as 3^2, which is 4.

22. A. Since the parabola has a vertical axis, it will be of the form $y = ax^2 + bx + c$. When we plug the points into this equation, we get the following system of equations:

$$\begin{aligned} 7 &= a \cdot 0^2 + b \cdot 0 + c \\ 15 &= a \cdot 4^2 + b \cdot 4 + c \\ 7 &= a \cdot 12^2 + b \cdot 12 + c \end{aligned}$$

and the solution to this system is $a = -\frac{1}{4}, b = 4, c = 7$, so the equation for the parabola is $y = -\frac{1}{4}x^2 + 3x + 7$ and the zeros or x-intercepts for this parabola are -2 and 14.

23. C. The slope of the radius from (0,0) to the point of tangency, $\left(\sqrt{3}, 2\right)$ is $\frac{2-0}{\sqrt{3}-0} = \frac{2}{\sqrt{3}}$. The tangent line will have a slope which is the negative reciprocal of this, so its slope is $-\frac{\sqrt{3}}{2}$. The slope from the desired y-intercept would be

$$-\frac{\sqrt{3}}{2} = \frac{2-b}{\sqrt{3}-0} = \frac{2-b}{\sqrt{3}} \Rightarrow 2(2-b) = -\sqrt{3}\cdot\sqrt{3} \Rightarrow 4-2b=-3 \Rightarrow 2b=7 \text{, so } b = 7/2.$$

24. E. The area will be $(12+2y)(12-0.5y) = 144+24y-6y-y^2 = 144+18y-y^2$, where y is the number of years. This quadratic expression has a maximum at its vertex, which occurs when $y = -\frac{b}{2a} = \frac{-18}{-2} = 9$. The area when $y = 9$ is $144+18(9)-9^2 = 225$.

25. D. If we look at all of the possible ways to get a 6, we have (1,5), (2,4), (3,3), (4,2), and (5,1). Two of these five have a 2.

26. A. If $\log(x+1)+2\log(x-1)=0 \Rightarrow \log\left((x+1)(x-1)^2\right)=0 \Rightarrow (x+1)(x-1)^2=1$, so we have $(x+1)(x-1)^2 = x^3-x^2-x+1=1 \Leftrightarrow x^3-x^2-x=0$. Now factoring yields $x\left(x^2-x-1\right)=0 \Rightarrow x=0, x=\dfrac{1\pm\sqrt{5}}{2}$, but only values greater than -1 can be used in the log expression.

27. B. The partial fraction problem has unique values for A and B.
$\dfrac{A}{x-1}+\dfrac{B}{x+2}=\dfrac{A(x+2)+B(x-1)}{(x-1)(x+2)}=\dfrac{(A+B)x+(2A-B)}{(x-1)(x+2)}=\dfrac{1x-3}{(x-1)(x+2)}$, so $A+B=1$ and $2A-B=-3$. We don't need to find A and B, since the question asks for A + B.

28. C. $12+64^x=8^{x+1} \Rightarrow 12+\left(8^x\right)^2=8^x\cdot 8 \Leftrightarrow w^2-8w+12=0$, where $w=8^x$, so $(w-6)(w-2)=0 \Rightarrow w=6,2 \therefore 8^x=2^{3x}=6,2 \Rightarrow 3x=\log_2 6 \Rightarrow x=\dfrac{\log_2 6}{3}$, or $2^{3x}=2 \Rightarrow x=\dfrac{1}{3}$.

29. B.
$$f(1-i)=\frac{(1-i)^2-(1-i)+1}{(1-i)+1}=\frac{(1-2i-1)-1+i+1}{2-i}=\frac{-i}{2-i}=\frac{-i(2+i)}{(2-i)(2+i)}=\frac{1-2i}{5}.$$

30. A. $f(1)=\dfrac{a}{1+b}=3$ and $f^{-1}(5)=1 \Rightarrow f(-1)=\dfrac{a}{-1+b}=5$, so $a=3+3b$ and $a=-5+5b$ and $3+3b=-5+5b \Rightarrow 8=2b \Rightarrow b=4$ and $a=3+3(4)=15$, making $f(0)=\dfrac{15}{0+4}$.

31. C. Since $\left(p^2-2p+3\right)=0$, we see that $\left(p^2-2p+3\right)^2=p^4-4p^3+10p^2-12p+9=0$, but we can rearrange the second expression to get $p^4-4p^3+10p^2-12p+9=p^4-4p^3+8p-2+\left(10p^2-20p+11\right)$, but this too can be rewritten as $\left[p^4-4p^3+8p-2\right]+10\left(p^2-2p+3\right)-19=0$, so the desired quantity $\left[p^4-4p^3+8p-2\right]=19-10\left(p^2-2p+3\right)=\boxed{19}$.

32. C. Subtracting the second equation from the first yields $y+2z=-1$. This equation subtracted from the third equation yields $z=6$, $y=-13$, and $x=\boxed{20}$.

33. D. The sequences of winning or losing are shown here:

WWWW: 24-32-40-48	WWWL:24-32-40-20	WWLW:24-32-16-24
WWLL:24-32-16-8	WLWW:24-12-20-28	WLWL:24-12-20-10
WLLW:24-12-6-14	WLLL:24-12-6-3	LWWW:8-16-24-32
LWWL:8-16-24-12	LWLW:8-16-8-16	LWLL:8-16-8-4
LLWW:8-4-12-20	LLWL:8-4-12-6	LLLW:8-4-2-10
LLLL:8-4-2-1		

As you can see, 9 of the 16 possibilities result in a final amount less than 16.

34. B.

$$\log_8(32x)-\log_4(8x)+\log_2(x)=\frac{\log_2(32x)}{\log_2 8}-\frac{\log_2(8x)}{\log_2 4}+\log_2(x)$$

$$=\frac{\log_2(32x)}{3}-\frac{\log_2(8x)}{2}+\log_2(x)=\frac{1}{6}\left(2\log_2(32x)-3\log_2(8x)+6\log_2 x\right)$$

$$=\frac{1}{6}\left(\log_2\left(\frac{(32x)^2\cdot x^6}{(8x)^3}\right)\right)=\frac{1}{6}\left(\log_2\left(\frac{2^{10}x^8}{2^9x^3}\right)\right)=\boxed{\frac{1}{6}\left(\log_2 2x^5\right)}$$

35. A. First we should find the two points. So $12-2x=5+6x-x^2$, which simplifies to $x^2-8x+7=0$. which factors to $(x-1)(x-7)=0 \Rightarrow x=1,7$ and the points of intersection are $(1,10)$ and $(7,-2)$. So the distance between these points is $\sqrt{(7-1)^2+(-2-10)^2}=\sqrt{6^2+(-12)^2}=\sqrt{36+144}=\sqrt{180}=\boxed{6\sqrt{5}}$.

36. C. IF we consider the numbers between 10 and 100 which have remainder of 3 when divided by 13, we get 16, 29, 42, 55, 68, 81, and 94. For each of these the remainder is 2, 1, 0, 6, 5, 4, and 3, respectively when divided by 7. We are looking for the one with remainder 3, so the desired number is 94. The product of its digits is 36.

37. B. In the figure, we have

$$\frac{5}{h}=\frac{25}{x}\Rightarrow x=5h \text{ and}$$

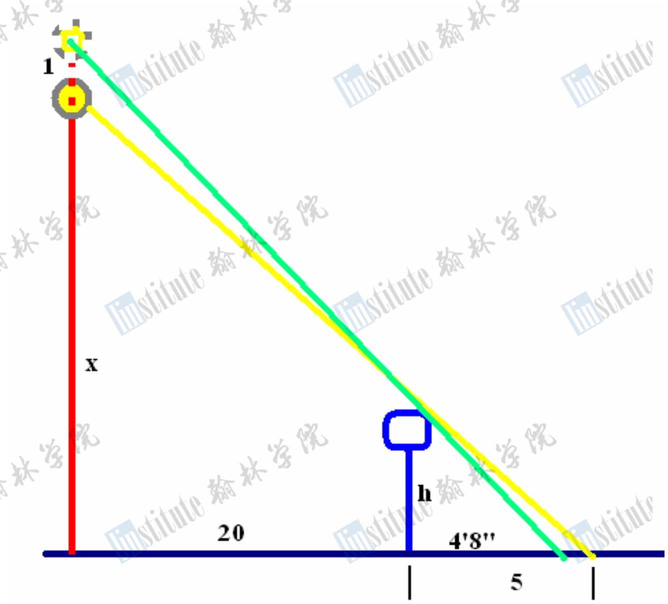
$$\frac{4'8''}{h}=\frac{24'8''}{x+1}\Leftrightarrow\frac{4\,^2\!/_3}{h}=\frac{24\,^2\!/_3}{x+1}\Leftrightarrow\frac{14}{h}=\frac{74}{x+1}.$$

Now, combining these we get

$$\frac{14}{h}=\frac{74}{5h+1}\Rightarrow 14(5h+1)=74h\Leftrightarrow$$

$$70h+14=74h\Rightarrow 14=4h\Rightarrow h=\frac{7}{2}, \text{ so}$$

$$x=5h=5\left(\frac{7}{2}\right)=\frac{35}{2}=17\,^1\!/_2\ ft.$$



38. E. If the angle of inclination is 30^0, the tangent of this angle is the slope, so the slope is $\tan(30^0)=\frac{\sqrt{3}}{3}$. Using the point-slope equation, we get $y-\sqrt{3}=\frac{\sqrt{3}}{3}(x-2)$. This simplifies to $y=\frac{\sqrt{3}}{3}(x-2)+\sqrt{3}$, which is equivalent to

$$y=\frac{\sqrt{3}}{3}(x-2)+\sqrt{3}=\frac{\sqrt{3}}{3}x-2\frac{\sqrt{3}}{3}+\frac{3\sqrt{3}}{3}=\frac{\sqrt{3}}{3}x-2\frac{\sqrt{3}}{3}+\frac{3\sqrt{3}}{3}=\frac{\sqrt{3}(x+1)}{3}$$

39. B. The only rational possibilities for roots are 1 and -1, and it is easy to show that both work. When we divide or factor these out, we have $f(x)=(x-1)(x+1)(x^2-6x-1)$. The remaining solutions are $x=\frac{6\pm\sqrt{40}}{2}=3\pm\sqrt{10}$. The larger of these is the largest root of the function, but -1 is the smallest, so the difference is $3+\sqrt{10}-(-1)=4+\sqrt{10}$..

40. A. Since the frame is only 2 pixels wide, it is composed of $2(2L)+2(2W)-16$. Note that the 16 pixels form the corners of the rectangle. So we have $4L+4W-16=504\Leftrightarrow 4L+4W=520$. If we increase the width by 10%, we have $2(2L\cdot 1.1)+2(2W)-16=524\Leftrightarrow 4.4L+4W=540$. Now subtract to find $0.4L=16\Rightarrow L=40$. From either of the two original equations, we see that $W=90$, and the image has $40(90)=3600$ pixels.