2005 State Mathematics Finals – Algebra I
Solutions

1. D. $\frac{x}{2}-\frac{x}{6}=\frac{3x}{6}-\frac{x}{6}=\frac{2x}{6}=\frac{x}{3}$, so x is a multiple of 3.

2. E. $7^{77}-7^{76}=7^{76}\left(7-1\right)=7^{76}\cdot 6$.

3. D. Average = $\frac{10\cdot 80+15\cdot 90}{25}=86$

4. E. $f(x+1)=3(x+1)^2+4=3\left(x^2+2x+1\right)+4=3x^2+6x+7$.

5. C. Let x be the first distance, y the second distance, and z the return distance. Using the Pythagorean theorem, $x^2=7^2+10^2\Rightarrow x=\sqrt{149}$, $y^2=4^2+5^2\Rightarrow y=\sqrt{41}$, and $z^2=14^2+12^2\Rightarrow z=\sqrt{340}$, so the total distance is $x+y+z=\sqrt{149}+\sqrt{41}+\sqrt{340}\approx 37.04$.

6. B. For the system of linear equation to contain infinitely many points, the lines must coincide, which happens when one equation is a scalar multiple of the other, so $ak=5, 3k=a, k=b$, which implies that $k=\frac{5}{a}=\frac{a}{3}=b$, so $a^2=15\Rightarrow a=\sqrt{15}$ and $b=\frac{\sqrt{15}}{3}\approx 1.29$

7. B. The linear regression equation with values rounded to 3 decimal places is $y=-68.616x+618.946$. When $x=7, y=-68.616(7)+618.946\approx \$138.63$.

8. A. $a=1.5c, b=1.25c\Rightarrow c=\frac{b}{1.25}$ and $a=1.5\left(\frac{b}{1.25}\right)=\frac{6}{5}b=1.2b$, so a is 20% larger than b.

9. D. Let the cost of living at the beginning of the year be x. At the end of the first quarter it is $1.02x$. At the end of the second quarter it increases another 2% to $\left(1.02\right)^2 x$. Continuing in this manner, at the end of four quarters the cost of living is $\left(1.02\right)^4 x=1.082x$. Thus, the increase of 2% per quarter corresponds to an annual increase of approximately 8.2%.

10. D. To find the last digit of $7^{42}+42^{7}$, we find the last digit of the two quantities in the sum. To find the last digit of 7^{42}, we notice that successive powers of 7 end in 7, 9, 3, 1, and then repeat in groups of 4, making the last digit of 7^{42} 9, since 42 has remainder 2 when divided by 4. Similarly, powers of 2 end in 2, 4, 8, 6, and back to 2, also cycling in groups of 4. Thus the last digit of 42^{7} is 8, since the remainder is 3. Thus the sum is the same as 9 + 8, or 7.

11. A. Since we want 12a3B to be divisible by 9, the sum of the digits, $A+B+6$, must be divisible by 9. Since this sum cannot equal 0, it must be 9 or 18, since 27 is too high even if A and B are both 9. So A + B = 3 or A + B = 12. Since we want 12A3B is to be divisible by 4, the number 3B formed by taking the last two digits must be divisible by 4. this can only be is B is 2 or 6. If B = 6, then by the restrictions on A + B, A must equal -3 (an impossibility) or 6, which is banned because A cannot equal B. This B = 2, so A = 1, or A = 10, another impossibility. The only solution is A = 1, B = 2.

12. C Let r = the number of rocks, s = the number of stones, p the number of pebbles. so $1r=7s$ and $1r=49p \Rightarrow 7s=49p \Rightarrow s=7p$, so $6r-(5r+2s+3p)=(5r+7s)-(5r+2s+3p)=5s-3p$, but we cannot have a negative number of coins, so $5s-3p=(4s+7p)-3p=4s+4p$.

13. E. Let a and $3a$ be the roots. Then $ma^2+8a+4=0$ and $m(3a)^2+8(3a)+4=0$, so $9ma^2+72a+36=0$ and $9ma^2+24a+4=0$. Subtracting the second from the first gives $48a+32=0$, so $a=-2/3$. Thus $m\left(-2/3\right)^2+8\left(-2/3\right)+4=0 \Rightarrow \frac{4}{9}m-\frac{16}{3}+4=0 \Rightarrow \frac{4}{9}m=\frac{4}{3} \Rightarrow m=3$.

14. B. $3^n+3^n=2\cdot 3^n$

15. D. $\frac{a-1}{a+1}=\frac{b-3}{b+3} \Rightarrow (a-1)(b+3)=(a+1)(b-4)$, so $ab+3a-b-3=ab-4a+b-4 \Rightarrow 7a+1=2b \Rightarrow b=\frac{7a+1}{2}$.

16. C. A rectangle with perimeter 24" has length = width equal to 12". Let x = width. Then $12-x=length$ and $(12-x)^2+x^2=10^2 \Rightarrow 144-24x+x^2+x^2=100$, so $2x^2-24x+44=0 \Rightarrow x^2-12x+22=0$, so $x(12-x)=22$, so Area = 22 sq. in.

17. A. $4x+y=10 \Rightarrow y=-4x+10$. If x is increased by 3, then $y'=-4(x+3)+10=-4x-2$, so y is decreased by 12 since 10 – (-2) = 12.

18. B. $\left(\frac{1}{2}\right)25000 = 25000(0.8)^x \Rightarrow \frac{1}{2} = 0.8^x \Rightarrow x \approx 3$.

19. B The set of possible scores are numbers of the form $5e+11h$, where e and h are nonnegative integers.

$e=0, 5e+11h=\{0,11,22,33,,...\}$

$e=1,\ 5e+11h=\{5,16,27,38,....\}$

For $e=2, 5e+11h=\{10,21,32,43,54,...\}$ Continuing in this manner, it can be

$e=3, 5e+11h=\{15,26,37,48,59,...\}$

$\vdots$

shown that 39 cannot be expressed in the form $5e+11h$, while the larger choices can. Note $53=5\cdot 4+11\cdot 3, 49=5\cdot 1+11\cdot 4, 43=5\cdot 2+11\cdot 3$.

20. E. $6^{x+y}=36 \Rightarrow 6^{x+y}=6^2 \Rightarrow x+y=2 \Rightarrow y=2-x$ and $6^{x+5y}=216 \Rightarrow 6^{x+5y}=6^3 \Rightarrow x+5y=3$. So $x+5(2-x)=3 \Rightarrow x+10-5x=3 \Rightarrow -4x=-7 \Rightarrow x=1.75$

21. C. $8^{x^2+3x+10}=4^{x^2-x} \Rightarrow 2^{3\left(x^2+3x+10\right)}=2^{2\left(x^2-x\right)} \Rightarrow 3\left(x^2+3x+10\right)=2\left(x^2-x\right)$, so $3x^2+9x+30=2x^2-2x \Rightarrow x^2+11x+30=0 \Rightarrow (x+5)(x+6)=0$, so $x=-5 or -6$, so the sum is $-5+(-6)=-11$. Note that the sum of the roots of the equation $x^2-bx+c=0$ is always b, so in this case, the sum is -11 without having to find the specific roots.

22. B. Since $x=(x-1)+1$, $f(x)=f(x-1+1)=(x-1)^2+3(x-1)+5=(x^2-2x+1)+3x-3+5=x^2+x+3$.

23. A. e for $n \neq -1, \dfrac{2\left(n^2+n\right)}{n+1}=\dfrac{2n(n+1)}{(n+1)}=\dfrac{2n}{1}=2n$, so $\langle 3x\rangle = 2\cdot(3x)=6x$.

24. D. $L@(L@K)=L@\left(L+\frac{K}{L}\right)=L+\left(\dfrac{L+\frac{K}{L}}{L}\right)=L+\dfrac{L}{L}+\dfrac{K}{L^2}=L+1+\dfrac{K}{L^2}$

25. E. Since n is even, $n=2k$, for some integer k. So $5n+4=5(2k)+4=2(5k+2)$, which must be divisible by 2.

26. B. Since Mark is y places in front of Sam, there are $x-y$ people behind Sam, excluding Sam. Then, since there are z people in front of Sam, again excluding Sam, the total number of people in line is $x-y+z+1$.

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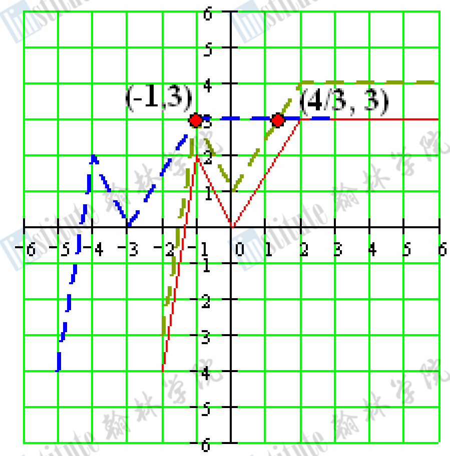
27. E. The question asks a graphical question: When (for what *x*-values) does the function shifted three units to the left equal the function shifted one unit up? The figure shows that $x=-1$ is one such point. The other point of intersection is two-thirds of the way up a segment that has a width of 2. Therefore, its horizontal change is 4/3. The segment begins at $x=0$, so $x=4/3$ is the other solution. We can check numerically.



28. A. The x and $2x+1$ be the width and length, respectively, of the original rectangle. Then the new rectangle has dimensions $x+5$ and $2x+6$. The areas are $x(2x+1)=2x^2+x$ and $(x+5)(2x+6)=2x^2+16x+30$, respectively, so the increase in area is $\left(2x^2+16x+30\right)-\left(2x^2+x\right)=15x+30$.

29. D. The slopes of the lines must be negative reciprocals.
$4y-x+3=0 \Rightarrow 4y=x-3 \Rightarrow y=\frac{1}{4}x-\frac{3}{4}$ and
$3y+ax+2=0 \Rightarrow 3y=-ax-2 \Rightarrow y=-\frac{a}{3}x-\frac{2}{3}$. So $\frac{1}{4}=-\left(-\frac{3}{a}\right) \Rightarrow a=12$.

30. E. If * is associative under some conditions, then $a*(b*c)=(a*b)*c$. Then $a*(b*c)=a*(c-2)=c-2-2=c-4$ and $(a*b)*c=(b-2)*c=c-2$. Since $c-4=c-2$, has no solution for *c*, there is no integer satisfying the desired condition and * is not associative.

31. E. The book prices are the terms of an arithmetic sequence where $a_1=x$, $a_2=x+2, a_3=x+2\cdot 2, \cdots a_n=x+(n-1)\cdot 2$. Either $a_{31}=a_{15}+a_{16}$ or $a_{31}=a_{16}+a_{17}$. If $a_{31}=a_{15}+a_{16}$, then $x+60=(x+30)+(x+28) \Rightarrow x=2$. If $a_{31}=a_{16}+a_{17}$, then $x+60=(x+30)+(x+32) \Rightarrow x=-2$, which is not possible for this situation. So the price of the cheapest book is $a_1=2$; the price of the middle book is $a_{16}=2+15\cdot 2=32$; and the price of the most expensive book is $a_{31}=2+30\cdot 2=62$, thus none of the statements are true.

32. E. If $a>b$, then $|a-b|=a-b$ and $|b-a|=-(b-a)=-b+a$. So $|a-b|+|b-a|=(a-b)+(-b+a)=2a-2b$.

33. B Look for a pattern: 6:38 to 6:40 gives three changes; 6:40 to 6:50 gives 11 changes, 6:50 to 7:00 gives twelve changes. The total at this point would be 26. Thus we need 1 more change. Hence 7:00 to 7:01.

34. D. Fifteen have two legs, six have three legs, and two have five legs. Jacob's solution: We start out by setting *x* to the number with two legs, *y* to the number with 3 legs, and *z* to the number with 5 legs. This process gives us these equations:

First day: $2x+3y+5z=58$

Second day: $4x+9y+25z=164$ To solve this system of equations, we will

Third day: $16x+81y+625z=1976$

use two pairs of equations to eliminate the *x*, then use the resulting pair of equations to eliminate the *y* and solve for *z*.

$$\left.\begin{array}{l}2(2x+3y+5z)=2\cdot 58\\ 4x+9y+25z=164\end{array}\right\}\Leftrightarrow\left.\begin{array}{l}4x+6y+10z=116\\ 4x+9y+25z=164\end{array}\right\}\Rightarrow 3y+15z=48$$

$$\left.\begin{array}{l}4(4x+9y+25z)=4\cdot 164\\ 16x+81y+625z=1976\end{array}\right\}\Leftrightarrow\left.\begin{array}{l}16x+36y+100z=656\\ 16x+81y+625z=1976\end{array}\right\}\Rightarrow 45y+525z=1320$$

So now we have the system

$$\left.\begin{array}{l}3y+15z=48\\ 45y+525z=1320\end{array}\right\}\Leftrightarrow\left.\begin{array}{l}15(3y+15z)=15\cdot 48\\ 45y+525z=1320\end{array}\right\}\Rightarrow\left.\begin{array}{l}45y+225z=720\\ 45y+525z=1320\end{array}\right\}\Rightarrow 300z=600$$

So $z=2,$ and substituting this into

$45y+525(2)=1320\Rightarrow 45y+1050=1320\Rightarrow 45y=270\Rightarrow y=6$. Now substituting the values for *y* and *z* into

$2x+3(6)+5(2)=58\Leftrightarrow 2x+18+10=58\Rightarrow 2x=30\Rightarrow x=15$, so

$x+y+z=15+6+2=23$.

35. C. Let *x* be the number of employees. Then

$\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=26\Rightarrow 12\left(\frac{x}{2}+\frac{x}{3}+\frac{x}{4}\right)=12(26)\Rightarrow 6x+4x+3x=312\Rightarrow 13x=312$, so

$x=24$.

36. A.

$\frac{4+20+x}{3}=\frac{y+16}{2}\Rightarrow 2(24+x)=3(y+16)\Rightarrow 48+2x=3y+48\Rightarrow 2x=3y$

o the ratio $\frac{y}{x}=\frac{3}{2}$.

37. A. There are 36 possible ways to roll two dice: (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),), (2,1), … (6,4), (5,4), (6,6). Of these possibilities, there are 9 rolls for

which the larger of the two numbers gives a score of 3 or less. So the probability of scoring a 3 or less is $\frac{9}{36}=\frac{1}{4}$.

38. A. $45x = 1350 \Rightarrow x = 40$.

39. E. $2f(x) = 2f\left(2\cdot\frac{x}{2}\right) = 2\left(\frac{2}{2+x/2}\right) = 2\left(\frac{4}{4+x}\right) = \frac{8}{4+x}$.

40. D. Since m and n must both be positive, it follows that $n>2$ and $m>4$. Because $\frac{4}{m}+\frac{2}{n}=1$ is equivalent to $(m-4)(n-2)=8$, we need only find all the ways of writing 8 as a product of positive integers. The 4 ways are $1\cdot 8, 2\cdot 4, 4\cdot 2, 8\cdot 1$ corresponds to 4 solutions, $(m,n)=(5,10),(6,6),(8,4),$ and $(12,3)$.