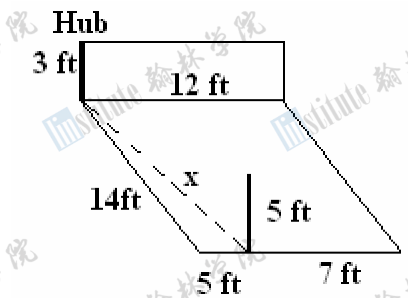
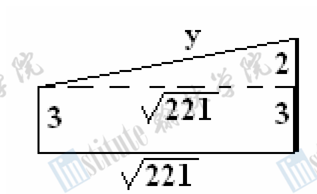
State Mathematics Finals 2004: Geometry Solutions

1. B) The ratio of the perimeters equals the ratio of the corresponding sides, bases, and heights, which equals 2:1. So the ratio of the areas is $2^2 : 1^2 = 4:1$.

2. A) Let $A = C = \frac{1}{2}B$. Then $A+B+C=180 \Rightarrow \frac{1}{2}B + B + \frac{1}{2}B = 180$,so $2B = 180 \Rightarrow B = 90$, but none of the angles are 90°. Let $A+C+2B$. Then $A+B+C=180 \Rightarrow 2B+B+2B=180$, so $5B=180 \Rightarrow B=36$. Then $A=C=2B \Rightarrow A=C=72$, and the smaller angle is 36°.

3. A) As shown in the figure, $x =$ distance on the floor from below the hub to below the desktop computer. Then $x^2 = 5^2 + 14^2 = 221$, so $x = \sqrt{221}$.

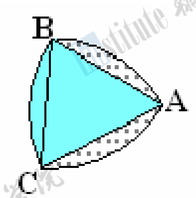




In the second figure, let y be the straight-line distance from the hub to the computer.

So $y^2 = 2^2 + \left(\sqrt{221}\right)^2 = 4 + 221 = 225$, so $y = \sqrt{225} = 15$.

4. D) The figure shown represents the intersection of the three overlapping circles. The shaded region show is a sector of a circle. Its area is one-sixth of the area of the circle with vertex A and radius AB. So the area is $\frac{1}{6} \cdot \pi r^2$. By subtracting the area of the equilateral triangle from the area just found, we can find the area of one of the dotted regions. $Area = \frac{1}{6}\pi r^2 - \frac{\sqrt{3}}{4} r^2$, so the area of the intersection is

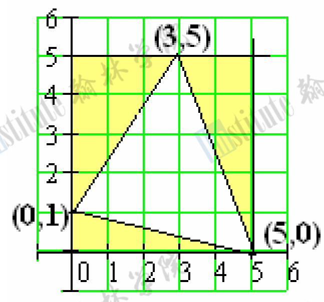


$$\frac{1}{6}\pi r^2 + 2\left(\frac{1}{6}\pi r^2 - \frac{\sqrt{3}}{4} r^2\right) = \frac{1}{2}\pi r^2 - \frac{\sqrt{3}}{2} r^2 = \left(\frac{\pi - \sqrt{3}}{2}\right) r^2.$$

5. A) Pat said Sam was a liar but Pat lies, so Same is not a liar, so what Sam says is true. Thus, either Pat, Chris, or Lour stole the cookie. Each one said they knew who ate the cookie. If Pat said this, since he lies, he does not know, so Pat could not have stolen the cookie, nor could Lou. Thus Chris stole the cookie.

6. E) Let r and $h$be the radius and height, respectively, of the original cyclinder. Then $2V = \pi(1.2r)^2(xh) \Rightarrow 2\pi r^2 h = \pi(1.44r^2)xh$, so $2 = 1.44x \Rightarrow x = 1.3\overline{8} = 138.\overline{8}\%$ of the original height so the increase is $38.\overline{8}\%$

7. C) $125 + 90 + 80 + x = 360 \Rightarrow x = 45^\circ$



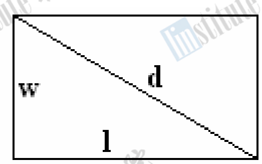
8. E) By plotting the points and construction a square, as shown., we can determine the area of the desired triangle by subtracting the areas of the three shaded right triangles from the area of the square. $A = 5^2 - \frac{1}{2}(5\cdot 1) - \frac{1}{2}(5\cdot 2) - \frac{1}{2}(4\cdot 3) = 11.5$ square units. Note too that is you know anything about determinants, the area is also

$$A = \left|\frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}\right| = \left|\frac{1}{2}\det\begin{bmatrix} 0 & 1 & 1 \\ 5 & 0 & 1 \\ 3 & 5 & 1 \end{bmatrix}\right| = \left|\frac{1}{2}(0+25+3)-(0+0+5)\right| = \frac{23}{2}$$

square units.

9. C) $A = l\cdot w + \frac{1}{2}\pi r^2 \Rightarrow 16000 = 80l + \frac{1}{2}\pi\cdot 40^2 \Rightarrow 16000 = 80l + 800\pi$, thus $l = 200 - 10\pi$. So the window height equals $l + 40 = 200 - 10\pi + 40 = (240 - 10\pi)$ cm.

10. D) Consider the rectangle as shown.



$P = 2l + 2w, A = l\cdot w, and\ l^2 + w^2 = d^2$ Thus

$P = 2l + 2w \Rightarrow P^2 = (2l + 2w)^2 = 4l^2 + 8lw + 4w^2$, So

$P^2 = 4(l^2 + w^2) + 8lw \Rightarrow P^2 = 4d^2 + 8A$, so

$P = \sqrt{4d^2 + 81} = 2\sqrt{d^2 + 2A}$

11. B) Since the measure of an exterior angle of a triangle equals the sum of the remote interior angles, for $\triangle ADB$, $81^\circ = m\angle D + m\angle B$, for $\triangle ADO$, $m\angle D + m\angle ABO = m\angle BAO$. Since $\triangle ADO$ and $\triangle ABO$ are isosceles, $m\angle D = m\angle AOD$ and $m\angle BOA = m\angle B$, so

$81^\circ = m\angle D + m\angle D, 2m\angle D = m\angle B \Rightarrow 81^\circ = m\angle D + 2m\angle D \Rightarrow m\angle D = 27^\circ$

12. E) The minute hand moves at a rate of $\frac{360^\circ}{1hr} = \frac{360^\circ}{60\min} = \frac{6^\circ}{\min}$. The hour hand moves at a rate of $\frac{360^\circ}{12hr} = \frac{30^\circ}{1hr} = \frac{30^\circ}{60\min} = \frac{1^\circ}{2\min}$. Let t = time, then

$$\frac{6^\circ}{1\min}\cdot t - \frac{1^\circ}{2\min}\cdot t = 90^\circ \Rightarrow 5.5^\circ t = 90^\circ \Rightarrow t = 16.\overline{36} = \frac{180}{11} = 16\frac{4}{11}\min.$$

13. A) The slope of the radius connecting (3,4) and (7,1) is $m = \frac{1-4}{7-3} = \frac{-3}{4}$, so the slope of the tangent through (7,1) is $\frac{4}{3}$. Thus

$$y - 1 = \frac{4}{3}(x-7) \Rightarrow 3y - 3 = 4x - 28 \Rightarrow 4x - 3y = 25.$$

14. C) The solution can be represented by the diagram where A, B, C, D, E, F, and G are the towns and "lines" or "curves are highways. ????????????

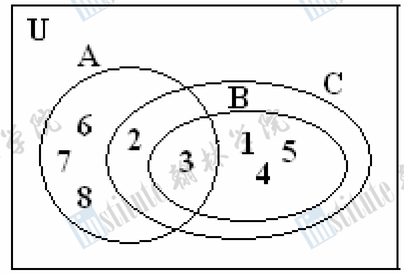
15. A) Let h be the height of the original pyramid and let c be the slant height. Then $h^2 + 6^2 = c^2 \Rightarrow c = \sqrt{h^2+36}$. In the new pyramid let $h+2$ be the height and c' the slant height. Then $c' = (h+2)^2 + 6^2 \Rightarrow c = \sqrt{h^2+4h+40}$. So the lateral surface area of the original pyramid is $4\left(\frac{1}{2}\cdot 12\cdot\sqrt{h^2+36}\right) or\ 24\sqrt{h^2+36}$.
So $24\sqrt{h^2+36} + 24 = 24\sqrt{h^2+4h+40} \Rightarrow \sqrt{h^2+36} + 1 = \sqrt{h^2+4h+40}$, so $h^2 + 36 + 2\sqrt{h^2+36} + 1 = h^2 + 4h + 40 \Rightarrow 2\sqrt{h^2+36} = 4h+3$. Squaring both sides yields $4(h^2+36) = 16h^2 + 12h + 9 \Rightarrow 12h^2 + 24h - 135 = 0$, or $4h^2 + 8h - 45 = 0$, so $h = \frac{-8\pm\sqrt{64-4(4)(-45)}}{8} = \frac{-8\pm\sqrt{784}}{8} = \frac{-8\pm 28}{8} = \frac{5}{2}, -\frac{9}{2}$, but only the positive value is possible.

16. E) The original triangle has perimeter $11+60+61=132$ units. The new triangle has sides x, 22, *and* $110-x$, where $110-x$ is the length of the hypotenuse. Then $x^2 + 22^2 = (110-x)^2 \Rightarrow x^2 + 484 = 12100 - 220x + x^2$, so $-11616 = -220x \Rightarrow x = 52.8 \Rightarrow 110 - x = 57.2$ units.

17. B) $(x-2)^2 + (y-1)^2 = 4 \Rightarrow x^2 - 4x + 4 + y^2 - 2y + 1 = 4$, and $(x-3)^2 + (y-4)^2 = 9 \Rightarrow x^2 - 6x + 9 + y^2 - 8y + 16 = 9$. Subtracting we get $2x + 6y = 15$, so this is the line through these points.

18. A) $m\angle DBA = m\angle OCA = 90^\circ$ since a tangent line is perpendicular to a radius drawn to the point of tangency. Then $m\angle BAC = 360^\circ - 2(90^\circ) - 144^\circ = 36^\circ$.

19. D) A Venn diagram can be constructed as shown to solve the problem. Note that $B = \{1,3,4,5\}$ and $A = \{2,3,6,7,8\}$, so the total of the values in A is $2+3+6+7+8=26$, which is twice the total of the value in B or $1+3+4+5=13$.

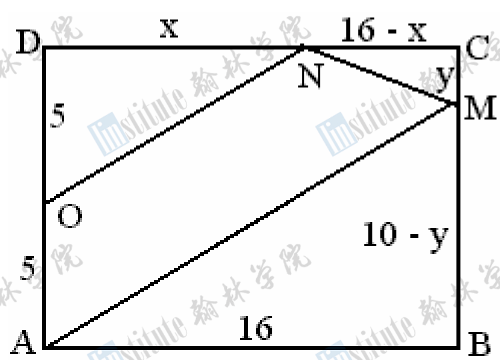


20. E) The distance the wheel travels is 4 minutes is distance = rate × time $= \left(\frac{30\,km}{hr}\cdot\frac{1\,hr}{60\min}\right)\cdot 4\min = 2\,km$, or 200,000 cm. A wheel with diameter 70 cm has circumference $\pi \cdot diameter = \pi(70cm) = 70\pi\ cm$, so in one revolution, the wheel turns $70\pi\ cm$.

21. A) The radius of the largest circle is $a+b$, so the area of the largest semicircle is $\pi(a+b)^2$. The areas of the two smaller semicircles are πa^2 and πb^2, so the area of the arbelos is $\frac{\pi(a+b)^2}{2} - \frac{\pi a^2}{2} - \frac{\pi b^2}{2} = \frac{2ab\pi}{2} = ab\pi$, but this is not one of our choices, so the segment with length h that is drawn, being an altitude to the hypotenuse of a right triangle (the one that can be inscribed in the semicircle) is the geometric mean between the segments of length $2a$ and $2b$, so $h = \sqrt{(2a)(2b)} = 2\sqrt{ab}$, so $ab = \frac{h^2}{4}$, making the area $\pi ab = \boxed{\frac{\pi h^2}{4}}$.

22. C) In a right triangle, the ratio of the side opposite one of the acute angles to the side adjacent to that angle is the tangent of the angle, so $\tan 3^\circ = \frac{h}{30} \Rightarrow h = 30 \cdot \tan 3^\circ \approx 1.572km = 1572m$. Adding this to the elevation of the observer makes the mountain 2205 m.

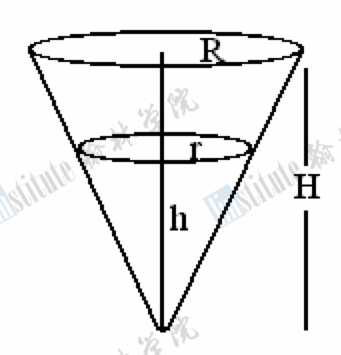
23. E) The triangles $\triangle ABM, \triangle NCM, \triangle NDO$ are all similar, so $\frac{5}{x} = \frac{y}{16-x} = \frac{10-y}{16}$. From this we have $xy = 80 - 5x, 16y = (16-x)(10-y)$. Solving for y in each of these gives $y = \frac{80-x}{x}$ and $y = \frac{160-10x}{32-x}$. So



$$\frac{80-5x}{x}=\frac{160-10x}{32-x}\Rightarrow 2x=32-x\Rightarrow 3x=32\Rightarrow x=10\tfrac{2}{3}.$$

24. D) We know that $a=21, c=b+3, A=\frac{1}{2}ab,\ and\ c^2=a^2+b^2$, so $(b+3)^2=21^2+b^2\Rightarrow 6b+9=441\Rightarrow 6b=432\Rightarrow b=72$. Thus the area is $\frac{1}{2}(21)(72)=756$.

25. E) The volume of a cone is $\frac{\pi}{3}r^2h$ and $\frac{r}{h}=\frac{R}{H}=\frac{6}{16}=\frac{3}{8}$, so $r=\frac{3}{8}h$. The original volume is $\frac{\pi}{3}\cdot 6^2\cdot 16=192\pi$. Four-fifths of this volume would result in a height of about 14.853, as follows:

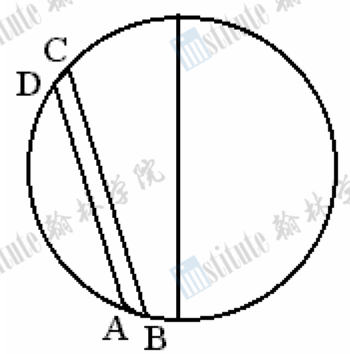


$\frac{4}{5}\cdot 192\pi=\frac{\pi}{3}\cdot r^2\cdot h=\frac{\pi}{3}\cdot\left(\frac{3}{8}h\right)^2\cdot h=\frac{3\pi}{64}h^3$, so

$h^3=\frac{64}{3}\cdot\frac{4}{5}\cdot 192\Rightarrow h\approx 14.853$. The percent change in the height is $\frac{16-h}{16}\approx 7.2\%$

26. B) The smallest angle is opposite the smallest side, so $\tan\theta=\frac{28}{45}\Rightarrow\theta=\tan^{-1}\left(\frac{28}{45}\right)\approx 31.9^\circ$

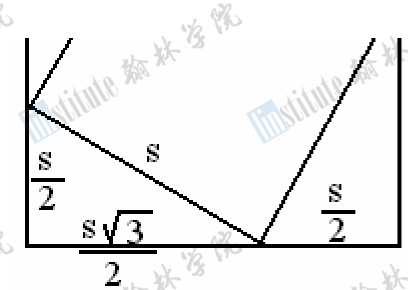
27. E) In the extreme case, if all four points are on the same half of the circle and A and B approach the same point while C and D approach the same point, BC and AD are both less than the diameter and AB and CD are close to zero, so proposition *y* is false while the other two are always true.



28. B) $S=\frac{s}{2}+\frac{s\sqrt{3}}{2}=s\left(\frac{1+\sqrt{3}}{2}\right)$, so

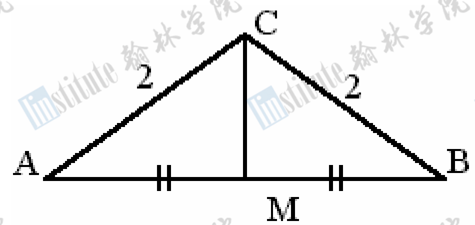
$S^2=s^2\left(\frac{4+2\sqrt{3}}{4}\right)\Rightarrow\frac{s^2}{S^2}=$

$\frac{4}{4+2\sqrt{3}}=4-2\sqrt{3}$



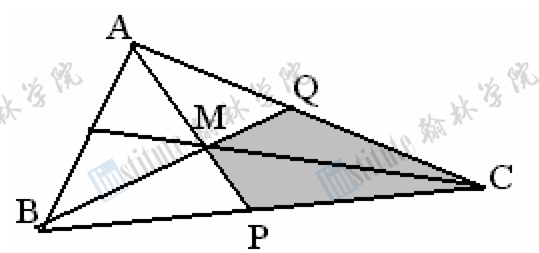
29. A) $a_m = 25m \text{ and } b_n = \frac{n(n+1)}{2}$, so $25m = \frac{n(n+1)}{2} \Leftrightarrow 50m = n(n+1)$. Now, since *m* and *n* are integers, we have $2\cdot 5^2 \cdot m = n(n+1)$, so it follows that 2*m* and 25 must be consecutive integers, making *m* = 12 and *n* = 24. so $\frac{m}{n} = \frac{12}{24} = 0.5$

30. E) Since $1+\sqrt{5} > 2$, and since the largest angle is opposite the longest side, we are looking for $\angle ACB$. (See Figure). Drop the perpendicular from C to M.



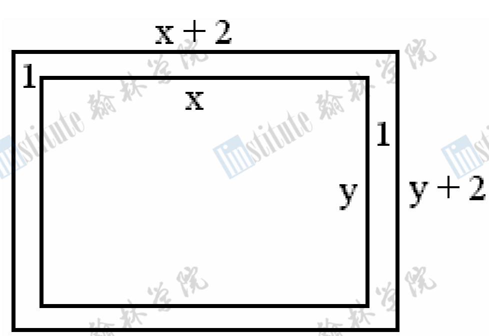
$\angle ACM = 2\angle MCB = 2\tan^{-1}\left(\frac{\left(1+\sqrt{5}\right)/2}{2}\right) = 108°$.

31. D) The medians of a triangle divide the triangle into 6 smaller, non-overapping triangles whose areas are each one-sixth of the area of the original triangle. The shaded region is made up of two of these regions, so the area is one-third of the area of the original.



32. B) Let the one angle have measure *x* and the other 2*x*. The third then is $(2x+x)-45$, so $x+2x+(3x-45)=180 \Rightarrow 6x=225$. The angles them measure 37.5, 75, and 67.5 degrees, making the largest 75 degrees.

33. C) The area of the larger rectangle is $(x+2)(y+2)$ and the area of the inner rectangle is *xy*. The border has area $(x+2)(y+2)-xy = 2x+2y+4$. Solving for *y* yields $y = \frac{2x+4}{x-2} = 2+\frac{8}{x-2}$. Since *x* and *y* must be integers, *x* could only be 3, 4, 6, or 10 making *y* 10, 6, 4, or 3, respectively. If we assume that the "smallest" rectangle is the one with the smallest area, then the combination of 4 and 6 gives an area of 24.

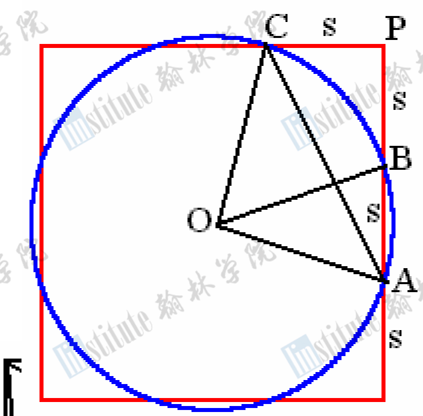


34. A) In the figure, draw the segment CA. In isosceles right triangle ACO, the length of AC is $10\sqrt{2}$. In triangle APC, with segments AB, BP, and PC all

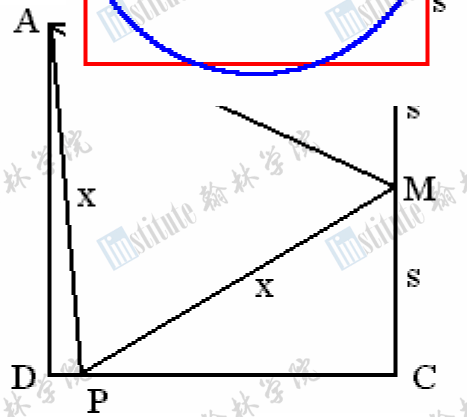
having length x, we have

$x^2+(2x)^2=\left(10\sqrt{2}\right)^2 \Leftrightarrow 5x^2=200 \Rightarrow x^2=40$.

The large square has area $(3x)^2=9x^2=360$.



35. B) Since triangles ABM and PCM are congruent, P must be the same point as D. Thus triangle ABM is a 30-60 right triangle and $BM=\frac{1}{2}AM$ and $AB=\frac{\sqrt{3}}{2}AB$, so $\frac{AB}{MB}=\sqrt{3} \Rightarrow \frac{AB}{BC}=\frac{\sqrt{3}}{2}$.



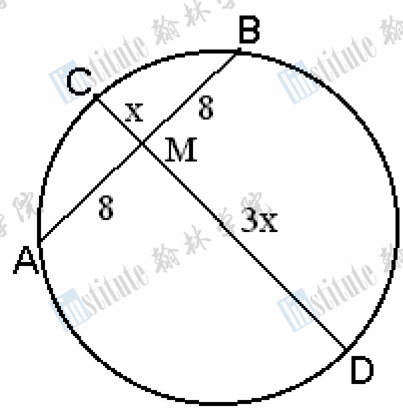
36. D) Since $192=0.75F=\frac{3}{4}F$, where F is the number of females, it follows that $F=\frac{4}{3}(192)=256$. So one-fourth of this number, 64, have not traveled outside the country. Thus the total in the survey is $256+64+184=504$.

37. B) The volume of the pipe, $V_{pipe}=2000cm\cdot\pi\cdot\left(\frac{1}{2}cm\right)^2=500\pi cm^3$. The time for this much water to be pushed through the pipe and replaced by hot water is $\frac{500\pi cm^3}{2800cm^3/\min}\cdot\frac{60\sec}{\min}\approx 33.66\sec$

38. D) By one of the "power of a point" theorems,

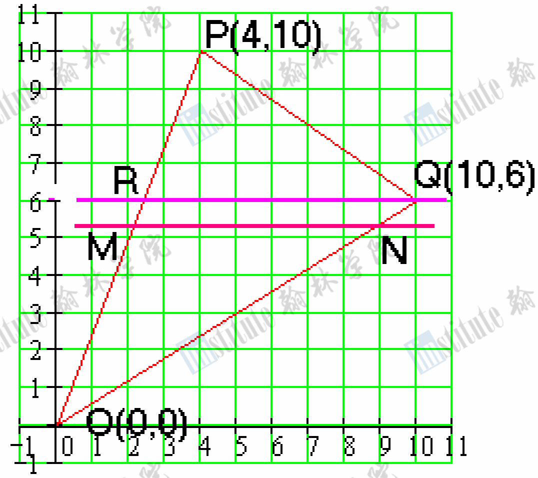
$x(3x)=8\cdot 8 \Rightarrow x^2=\frac{64}{3} \Rightarrow x=\frac{8}{\sqrt{3}}=\frac{8\sqrt{3}}{3}$, so

$CD=4x=\frac{32\sqrt{3}}{3}$.



39. A) First notice that when $y=6$, the top triangle PRQ has a vertical altitude of 4 and since the equation of line OP is $y=\frac{2}{5}x$, when

$y = 6,\ x = \frac{12}{5}$, so the base RQ has length $10 - \frac{12}{5} = \frac{38}{5}$. Thus the area is $\frac{1}{2} \cdot \frac{38}{5} \cdot 4 = \frac{76}{5}$, but this less than half the area of triangle OPQ. A quick way to find the area of triangle OPQ is the determinant formula



$$A = \left| \frac{1}{2} \cdot \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right| = \left| \frac{1}{2} \cdot \begin{vmatrix} 0 & 0 & 1 \\ 10 & 6 & 1 \\ 4 & 10 & 1 \end{vmatrix} \right| = \left| \frac{1}{2}(10-24) \right| = 38.$$ So the horizontal line will hit the lines OQ and OP. The altitude will be the y-value and the horizontal base will be the distance between the lines OP and OQ along the horizontal line. The coordinates of the points M and N, in terms of y are $\left(\frac{2}{5}y, y\right)$ and $\left(\frac{5}{3}y, y\right)$, so the base is $\frac{5}{3}y - \frac{2}{5}y = \frac{19}{15}y$, so the area is $\frac{1}{2}\left(\frac{19}{15}y\right)y = \frac{19y^2}{30}$. This must be half the area of the original triangle, so $\frac{19y^2}{30} = \frac{38}{2} \Rightarrow 38y^2 = 30 \cdot 38 \Rightarrow y^2 = 30$, so $y = \sqrt{30}$. (Need Figure)

40. B) It takes 3 revolutions of the crank wheel to turn the larger wheel one time. In one revolution of the larger wheel, the spool will also make one revolution. For each revolution of the spool, the length of rope holding up the picture shortens by one circumference of that spool, or $12\pi\, in$. For each unit the rope shortens, the picture moves up by half a unit, so turning the crank 5 revolutions results in five-thirds revolutions, so the string shorted by $\frac{5}{3} \cdot 12\pi = 20\pi$ and the picture goes up by half that or 10π.