Mathematics Contest Spring 2004

Solutions and $\boxed{\text{Answers}}$

Part I: Multiple Choice (20 Problems)

1. When $\dfrac{5x^2+20x+6}{x^3+2x^2+x}$ is decomposed into partial fractions, with each term reduced to lowest terms, the sum of the numerators is

a. 16 b. 15 $\boxed{\text{c. 14}}$ d. 12 e.-4

SOLUTION: $\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}=\dfrac{Ax^2+2Ax+A+Bx^2+Bx+Cx}{x(x+1)^2}=\dfrac{5x^2+20x+6}{x^3+2x^2+x}$

gives A = 6, B = -1 and C = 9 so A+B+C = 14

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2 A person deposits \$500 into a savings account at the end of every month for 4 years at 6% annual rate compounded monthly. How much interest will be earned during the 4 years?

a. \$1440 b.\$1480.27 c. \$2024.39 $\boxed{\text{d. \$3048.92}}$ e. \$4098.46

SOLUTION: *The interest on \$500 after k months is $500(1.005^k-1)$. The total interest is $500\{1.005^{47}-1+1.005^{46}-1+\ldots+1.005^0-1\}=500\left(\dfrac{1.005^{48}-1}{1.005-1}-48\right)=3048.92$*

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3. There are 100 members of the senate, 2 from each state. In how many ways can a committee of 5 senators be formed if no state may be represented more than once?

a. 2,118,760 b. 75,287,520 c. 4,950 $\boxed{\text{d. 67,800,320}}$ e. 254,251,200

SOLUTION: *Choose five of fifty states, $\binom{50}{5}$ ways, and one of two senators from each of the five state, 2^5 ways.* $\binom{50}{5}2^5=67{,}800{,}320$

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4. You have 6 sticks of lengths 10, 20, 30, 40, 50, and 60 centimeters. The number of non-congruent triangles that can be formed by choosing three of the sticks to make the sides is

a. 3 b. 6 $\boxed{\text{c. 7}}$ d. 10 e. 12

SOLUTION: *By the triangle inequality, the possible lengths of sticks that form triangles are (20,30,40), (20,40,50), (20,50,60), (30,40,50), (30,40,60), (30,50,60), (40,50,60).*

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5. A glass box $7\text{ cm} \times 12\text{ cm} \times 18\text{ cm}$, closed on all six sides is partly filled with colored water. When the box is placed on one of its 7×12 sides the water level is 15 cm above the table. When the box is placed on one of its 7×18 sides the water level above the table, in centimeters, will be

a. 7.5 b. 9 $\boxed{\text{c. 10}}$ d. 12.5 e. none of these

SOLUTION: *If the water is x cm above the table when the box is placed on the 7×18 side, then the volume of water = $7 \times 12 \times 15 = 7 \times 18 \times x$ so $x=10$*

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6. Two integers are said to be partners if both are divisible by the same set of prime numbers. The number of positive integers less than 25 that have no partners less than 25 is

a. 11 $\boxed{\text{b. 12}}$ c. 13 d. 16 e. 24

SOLUTION: *Integers without partners are 1,5,7,11,13,14,15,17,19,21,22,23*

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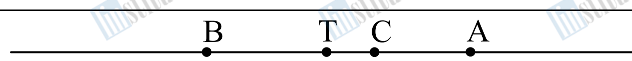
7. There are four cottages on a straight road. The distance between Ted's and Alice's cottages is 3 kilometers. Both Bob's and Carol's cottages are twice as far from Alice's as they are from Ted's. In kilometers, the distance between Bob's and Carol's cottages is

a. 1 b. 2 c. 3 $\boxed{\text{d. 4}}$ e. 6

SOLUTION:



As shown above, without loss of generality, we may assume Carol lives between Alice and Ted, one km. from Ted and two km. from Alice's. Bob's cottage is three km. from Ted, on the other side from Alice and six km. from Alice. Since Bob's cottage is three km. from Ted's and Carol's is 1 km. from Ted's, the distance between Bob's and Carol's cottages is 4 km.

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8. Al, Bee, Cecil, and Di have \$16, \$24, \$32, and \$48 respectively. Their father proposed that Al and Bee share their wealth equally, and then Bee and Cecil do likewise and then Cecil and Di. Their mother's plan is the same except that Di and Cecil begin by sharing equally, then Cecil and Bee and then Bee and Al. The number of children who end up with more money under their father's plan than under their mother's is

a. 0 b. 1 $\boxed{\text{c. 2}}$ d. 3 e. 4

SOLUTION: *Father's plan: Al will have* $\frac{16+24}{2} = \$20$, *Bee will have* $\frac{20+32}{2} = \$26$

Cecil will have $\frac{26+48}{2} = \$37$ *and Di will also have \$37.*

Mother's plan: Di will have $\frac{48+32}{2} = \$40$, *Cecil will have* $\frac{40+24}{2} = \$32$, *Bee and Al will both have* $\frac{32+16}{2} = \$24$ *So Bee & Cecil will end up with more money under father's plan.*

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9. Let $t_0 = 2004$ and recursively define $t_{k+1} = [\frac{1}{2}(t_0-t_1-t_2-\ldots-t_k)]$ where $[x]$ is the greatest integer less than or equal to x. Find the least number k so that $t_k = 0$.

a. 9 b. 10 c. 11 $\boxed{\text{d. 12}}$ e. t_k is never 0

SOLUTION: *Consider the integers* $s_k = t_0-t_1-t_2-\ldots-t_k$ *for* $k \ge 0$.

By definition $t_{k+1} = [\frac{1}{2}s_k] = \begin{cases} \frac{1}{2}s_k & \text{if } s_k \text{ even} \\ \frac{1}{2}(s_k-1) & \text{if } s_k \text{ odd} \end{cases}$

so $s_{k+1} = s_k - t_{k+1} = \begin{cases} \frac{1}{2}s_k & \text{if } s_k \text{ even} \\ \frac{1}{2}(s_k+1) & \text{if } s_k \text{ odd} \end{cases}$

Assume that $n < s_k \le 2n$. *If* s_k *is even, then* $\frac{1}{2}n < s_{k+1} \le n$.

If s_k *is odd, then* $s_k+1 \le 2n$ *and so also* $\frac{1}{2}n < s_{k+1} \le n$.

Since $2^{10} < 2004 \le 2^{11}$, *applying this result 10 times yields* $1 < s_{10} \le 2$.
Thus $s_{10} = 2$, $t_{11} = [\frac{1}{2}s_{10}] = 1$ *and* $t_{12} = 0$.

The sequence s_k *is {2004, 1002, 501, 251, 126, 63, 32, 16, 8, 4, 2} for k=0, ….,10 and the sequence* t_k *is {2004, 1002, 501, 250, 125, 63, 31, 16, 8, 4, 2, 1, 0} for k=0,…,12*

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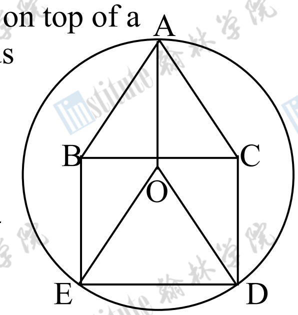
10. A pentagon is made up of an equilateral triangle ABC of side length 2 on top of a square BCDE. Circumscribe a circle through points A, D and E. The radius of the circle is:



a) $\frac{\sqrt{3}}{2}+1$ $\boxed{\text{b) 2}}$ c) $\sqrt{3}+1$ d) $5-2\sqrt{3}$ e) $\sqrt{2}$

SOLUTION: *Consider the equilateral triangle EDO with O inside the square. Then ABEO is a parallelogram with each side of equal length. Thus O is the center of the circle and the radius is 2.*

11. Two of the roots of the equation $2x^3 - 3x^2 + px + q = 0$ are 3 and -2. The third root is

$\boxed{\text{a. } \frac{1}{2}}$ b. $\frac{-5}{2}$ c. -3 d. $\frac{1}{3}$ e. 1

SOLUTION: *The sum of the roots is the negative of the coefficient of x^2 of the associated monic polynomial, i.e.* $\frac{3}{2}$. *Thus the third root is* $\frac{1}{2}$.

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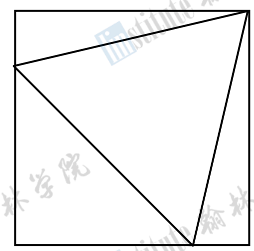
12. How long is the side of the largest equilateral triangle that can be inscribed in a square whose side has length 1?

a. 1 b. $\frac{\sqrt{5}}{2}$ c. $\frac{3\sqrt{5}}{4}$ d. $2-\sqrt{3}$ $\boxed{\text{e. } \sqrt{8-4\sqrt{3}}}$

SOLUTION: *By symmetry, the largest inscribed equilateral triangle has one vertex at a vertex of the square and adjacent sides making angle of* $\frac{1}{2}(90^o - 60^0) = 15^o$. *So the length of*



triangle's side $= \dfrac{1}{\cos 15^o} = \dfrac{1}{\sqrt{\dfrac{1+\cos 30^o}{2}}} = \sqrt{\dfrac{2}{1+\dfrac{\sqrt{3}}{2}}} = \dfrac{2\sqrt{2-\sqrt{3}}}{\sqrt{2^2-3}}$

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13. A round table can be made square by dropping the four leaves. If a side of the square table measures 36 inches, approximately how much smaller is the area of the table when the leaves are down than when the leaves are up?

$\boxed{\text{a. 750 in}^2}$ b. 850 in^2 c. 1000 in^2 d. 1250 in^2 e. 1300 in^2

SOLUTION: *The diameter of the circle = diagonal of the square =* $36\sqrt{2}$.

So the difference in areas is $\pi\left(\dfrac{36\sqrt{2}}{2}\right)^2 - 36^2 \simeq 739.75$

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14. From a fixed point on a circle chords are drawn to the other points on the circle. What is the locus of midpoints of the chords?

a. non-circular ellipse $\boxed{\text{b. circle}}$ c. hyperbola d. parabola e. line

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SOLUTION: *Let the circle of radius c be tangent to the y-axis with the fixed point at the origin. Then the equation of the circle in polar coordinates is r=2c*cos* θ. *The locus of midpoints of this circle has the equation r = c*cos* θ *and so is a circle of radius* $\dfrac{c}{2}$.

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15. The sum of the two largest numbers x for which the determinant

$$\begin{vmatrix} 2x-2 & 1 & 4 \\ 6x-11 & 2x-5 & 2x+5 \\ -2x+2 & -1 & x-2 \end{vmatrix}$$ equals zero is

a. 20 $\boxed{\text{b. } 5}$ c. 2   d. $\frac{-1}{2}$ e. none of the above

***SOLUTION**: To simplify the calculation, first add the first row to the third. This does not change the value of the determinant. Det = (x+2)(4x^2 – 20x + 21) The linear factor has a negative zero and the quadratic factor has two positive zeros that sum to $\frac{20}{4}$ = 5.*

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16. Consider the circles with radii $4\sqrt{5}$ and which are tangent to the line $x - 2y = 20$ at the point (6, -7). The sum of the x coordinates of the centers of the circles is

$\boxed{\text{a. } 12}$ b. -14   c. 3   d. -5   e. 2

***SOLUTION**: The centers will be symmetrically placed along a line through (6, -7) perpendicular to x-2y=20. Thus the average values of the x coordinates will be 6.*

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17. Given the equation $x^3 - 2x^2 + x - 3 = 0$, an equation whose roots are each 2 less than the roots of the given equation is

a. $x^3 - 8x^2 + 21x - 21 = 0$ b. $x^3 - 4x^2 - x - 5 = 0$ c. $x^3 - 4x^2 + 2x - 6 = 0$

$\boxed{\text{d. } x^3 + 4x^2 + 5x - 1 = 0}$ e. $x^3 + 4x^2 - 2x + 6 = 0$

***SOLUTION**: $(x+2)^3 - 2(x+2)^2 + (x+2) - 3 = x^3 + 4x^2 + 5x - 1$*

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18. An experiment consists of choosing with replacement an integer at random among the numbers from 1 to 9 inclusive. If we let M denote a number that is an integral multiple of 3 and N denote a number that is not an integral multiple of 3, which of the following sequences of results is least likely?

a. M N N M N   b. N M M N $\boxed{\text{c. N M M N M}}$ d. N N M N   e. M N M M

***SOLUTION**: Since there are 3 multiples of 3 among the digits 1 to 9, the probability that a digit is of type M is $\frac{1}{3}$ while the probability that it is of type N is $\frac{2}{3}$*

The probability of each of the sequences occurring is

Pr(MNNMN) = $\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = \frac{8}{243}$,   *Pr(NMMN)*= $\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 = \frac{4}{81}$,

Pr(NMMNM) = $\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 = \frac{4}{243}$,   *Pr(NNMN)* = $\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3 = \frac{8}{81}$,

Pr(MNMM)= $\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right) = \frac{2}{81}$.

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19. An 8 foot by 8 foot area has been tiled by one foot square tiles. Two of the tiles were defective. What is the probability that the two defective tiles share an edge?

a. $\frac{1}{8}$ b. $\frac{1}{12}$ c. $\frac{1}{16}$ d. $\boxed{\frac{1}{18}}$ e. $\frac{1}{64}$

SOLUTION: *There are* $\binom{8^2}{2} = \frac{8^2(8^2-1)}{2}$ *ways of choosing two arbitrary squares to place the two defective tiles.*

If the two defective tiles share an edge, then

Case i: One of the tiles was placed in any of the top 7 rows (8×7 ways), the other was placed in the square below.

Case ii: One tile was placed in any of the left 7 columns (8×7 ways), the other was placed in the square to its right.

So the probability is $\frac{2\times 8\times 7}{\frac{64\times 63}{2}} = \frac{1}{18}$

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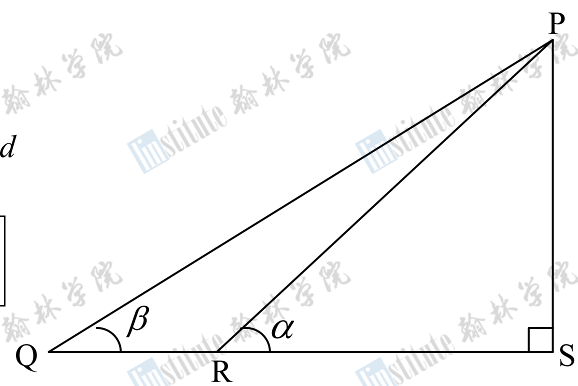
20. In the diagram if QR = d, then PS equals



a. $\frac{sin(\beta)}{sin(\alpha-\beta)}d$ b. $\frac{tan(\beta)}{tan(\alpha)-tan(\beta)}d$

c. $\frac{d}{tan(\alpha)-tan(\beta)}$ d. $\boxed{\frac{d}{cot(\beta)-cot(\alpha)}}$

e. $\frac{sin(\alpha)sin(\beta)}{cos(\alpha)(sin(\alpha)-sin(\beta))}d$

SOLUTION: *Let x=PS. Then x=(d +x cot(α)) tan(β) . Solving for*

$$x = \frac{d\,tan(\beta)}{1-cot(\alpha)tan(\beta)} = \frac{d}{cot(\beta)-cot(\alpha)}$$

Part II: Integer Answer (15 Problems)

1. Find n so that $\dfrac{1}{1+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{5}}+\dfrac{1}{\sqrt{5}+\sqrt{7}}+\ldots+\dfrac{1}{\sqrt{2n-1}+\sqrt{2n+1}}=100$

***SOLUTION**: After rationalizing denominators, the left hand side becomes*

$$\frac{1-\sqrt{3}}{1-3}+\frac{\sqrt{3}-\sqrt{5}}{3-5}+\frac{\sqrt{5}-\sqrt{7}}{5-7}+\cdots+\frac{\sqrt{2n-1}-\sqrt{2n+1}}{(2n-1)-(2n+1)}=\frac{1-\sqrt{2n+1}}{-2}$$

Solving $\dfrac{1-\sqrt{2n+1}}{-2}=100$ *we get* $2n+1 = (200+1)^2$ *So* $n = \boxed{20200.}$

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2. If **tan 3*x*** is written in terms of **tan *x***, $\tan 3x=\dfrac{A\tan x-B\tan^3 x}{1-C\tan^2 x}$ find $\mathrm{A+B+C}$.

SOLUTION: *As x approaches* $\dfrac{\pi}{6}$, $\tan^2 x$ *approaches* $\frac{1}{3}$ *while tan 3x approaches* ∞.

Thus C=3.

If $x = \dfrac{\pi}{3}$, $0 = \dfrac{\sqrt{3}A-3\sqrt{3}B}{1-3C}$ *so A=3B.*

If $x = \dfrac{\pi}{4}$, *then* $-1 = \tan\dfrac{3\pi}{4} = \dfrac{A-B}{1-C}$ *so A-B=2, A=3, B=1, C=3 and A+B+C =* $\boxed{7.}$

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3. Consider the equation $15x+14y=7$. Find the largest four digit integer x for which there is an integer y so that the pair (x, y) is a solution.

SOLUTION: *15(7)+14(-7)=7 so (7, -7) is a solution.*

For x=7+s to be a solution, $y=\dfrac{7-15x}{14}=\dfrac{7-15(7+s)}{14}=-7-\dfrac{15s}{14}$ *must be an integer,*

so 14 must divide 15s. Since 14 and 15 are relatively prime, s must be a multiple of 14 thus s=14t for some integer t and x=7+14t.

If x=7+14t $\leq$ *9999, then* $t\leq\dfrac{9992}{14}\simeq 713.7$ *So the largest t=713 and x=*$\boxed{9989.}$

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4. Let P be the set of primes that divide 200! (i.e. 200 factorial). What is the largest integer k, so that the set of primes that divides k! is equal to P?

***SOLUTION**: P = {p prime, p$\leq$ 200} = { 2, 3, . . . , 197}. Since 211 is the smallest prime greater than 200, these are the only primes that will divide all integers $\leq$ k! for k =* $\boxed{210.}$

5. What is the remainder when $7^{348}+25^{605}$ is divided by 8?

SOLUTION: $7^2 \equiv 1\,(mod\,8)$ *so* $7^{348}=(7^2)^{174}\equiv 1^{174}=1\,(mod\,8)$.

$25\equiv 1\,(\mathrm{mod}\,8)$ *so* $25^{605}\equiv 1^{605}=1\,(\mathrm{mod}\,8)$.

Thus $7^{348}+25^{605}\equiv 1+1=2(mod\,8)$. *The remainder is* $\boxed{2.}$

6. How many possible values can there be for three coins selected from among pennies, nickels, dimes and quarters?

SOLUTION: *Case i, all three are of the same denomination: 4 ways*

Case ii, two coins of any one of the four denominations, the third of any of the remaining three denominations: $4\times 3=12$ *ways.*

Case iii, all three are of different denominations: $\binom{4}{3}$ *=4 ways.*

Adding up the number of ways for all three cases, we get 4+12+4=$\boxed{20}$ *ways.*
One checks that the values given by these 20 ways are all different.

7. A water tank has been sanitized by pouring in chlorine bleach. Bleach is toxic at the level needed to sanitize, so you need to flush out the tank using clean water. The result is that after each hour of flushing there is a 19% reduction in the bleach concentration. Assume that when you began flushing, the bleach concentration is 150 mg/gal. You can safely use the water tank for drinking purposes when the bleach concentration is below 0.7 mg/gal. What is the minimum number of whole hours you should flush the tank for safe drinking purposes?

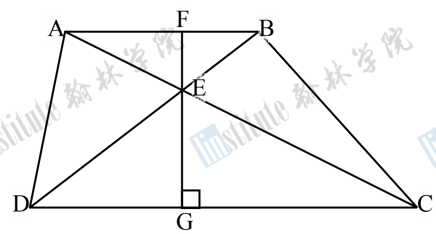
SOLUTION: *After the first hour of flushing, 81% of the 150 mg/gal, or 150(0.81) mg/gal will be left, after k hours,* $150(0.81)^k$ *mg/gal will be left.*

If $150(0.81)^k \le 0.7$, $k\,ln(0.81)\le ln(0.7/150)$ *so* $k \ge \dfrac{ln(0.7/150)}{ln(0.81)} \simeq 25.47$

and $k=\boxed{26}$ *hours.*

8. In a trapezoid ABCD with AB parallel to CD, the diagonals intersect at point E. The area of triangle ABE is 32 and of triangle CDE is 50. Find the area of the trapezoid.

SOLUTION: *Since AB is parallel to CD,* ΔEAB *is similar to* ΔECD. *Let FEG be the line through E perpendicular to AB & CD, intersecting AB at F and CD at G*



Let $\dfrac{AB}{CD}=\dfrac{FE}{GE}=r$ *so* $64 = 2\,|\Delta EAB| = |AB|\,|FE| =$

$r^2\,|CD|\,|GE| = 2r^2\,|\Delta ECD| = 100r^2$. *Thus r = 0.8*

$|ABCD| = \frac{1}{2}\,|GF|\,(|AB| + |CD|) = \frac{1}{2}(1.8\,|GE|)\,(1.8\,|CD|) = 1.8^2\,|\Delta ECD| = \boxed{162.}$

9. Find the number of 4 digit positive integers which are divisible by 3 and/or 7.

SOLUTION: *Let [x] be the largest integer less than or equal to x. Then there are* $\left\lfloor \frac{9999}{n} \right\rfloor$ *positive integers* ≤ 9999 *divisible by n, of which* $\left\lfloor \frac{999}{n} \right\rfloor$ *of them are* ≤ 999 *so*

there are $\left\lfloor \frac{9999}{n} \right\rfloor - \left\lfloor \frac{999}{n} \right\rfloor$ *4 digit positive integers divisible by n.*

Letting n=3, there are $\left\lfloor \frac{9999}{3} \right\rfloor - \left\lfloor \frac{999}{3} \right\rfloor = 3000$ *four digit positive integers divisible by 3.*

Similarly there are $\left\lfloor \frac{9999}{7} \right\rfloor - \left\lfloor \frac{999}{7} \right\rfloor = 1286$ *divisible by 7. This includes*

$\left\lfloor \frac{9999}{21} \right\rfloor - \left\lfloor \frac{999}{21} \right\rfloor = 429$ *that are divisible by both 3 and 7 and so are counted twice.*

So there are $3000+1286-429 = \boxed{3857}$ *4 digit positive integers divisible by 3 and/or 7.*

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10. What is the smallest positive integer which when divided by 10, 9, 8, 7, 6 leaves the remainder 9, 8, 7, 6, 5 respectively?

SOLUTION: *Let n be the integer to be found. Since* $n = 10x+9 = 10(x+1) - 1$ *so n+1 is a multiple of* $10=2(5)$. *Similarly, n+1 is a multiple of* $9=3(3)$, $8=2^3$, 7, $6=2(3)$. *So n+1 must contain the factors* 2^3, 3^2, 5, 7. *Thus* $n+1=2^3 3^2(5)(7) = 2520$ *and* $n=\boxed{2519.}$

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11. If the product of three numbers in geometric progression is 216 and their sum is 19, then the largest of the three numbers is

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SOLUTION: *Let the 3 numbers be* a, ax, ax^2 *for some a & x.*

$a(ax)(ax^2) = a^3x^3 = 216 = 6^3$ *so* $ax = 6$ *and* $a = \frac{6}{x}$

$a+ax+ax^2 = a(1+x+x^2) = 19$, *substituting* $a = \frac{6}{x}$, *we get* $6+6x+6x^2 = 19x$

$6x^2-13x+6 = (3x-2)(2x-3) = 0$ *or* $x = \frac{2}{3}$ *or* $\frac{3}{2}$

if $x = \frac{2}{3}$, *then* $a = \frac{6}{x} = 9$ *and the numbers are 9, 6, 4*

if $x = \frac{3}{2}$, *then* $a = \frac{6}{x} = 4$ *and the numbers are 4, 6, 9*

So the largest of the three numbers is $\boxed{9.}$

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12. Among all collections of positive integers whose sum is 28, what is the largest product that the integers in S can form?

SOLUTION: *If $n \geq 4$ is in S, then $2(n-2) \geq n$. Thus the largest product with a given sum occurs with no integers larger than 3. Since $2^3 < 3^2$, two 3's give a larger product than three 2's. Since $3 \times 1 < 2^2$, we should choose two 2's instead of a 3 and a 1.*

$28 = 3(8)+2(2)$ and the largest product is $3^8 2^2 = \boxed{26244}$.

13. Consider the set S of positive integers d for which there exists an integer n such that d evenly divides both $(13n+6)$ and $(12n+5)$. Then the sum of the elements of S is

SOLUTION: *Since d divides both 13n+6 and 12n+5, d divides*

$12(13n+6) - 13(12n+5) = 7$ so d can only be 1 or 7.

If $n = 6$, $12n+5 = 77$ and $13n+6 = 84$ are both divisible by 1 and 7.

Thus S = {1,7} and its sum is $\boxed{8}$.

14. Suppose that x and y are two real numbers such that $x - y = 2$ and $x^2 + y^2 = 8$. Find $x^3 - y^3$.

SOLUTION: $2^2 = (x-y)^2 = x^2 - 2xy + y^2 = 8 - 2xy$, so $xy = 2$

Thus $x^3 - y^3 = (x-y)(x^2+xy+y^2) = 2(8+2) = \boxed{20}$.

15. What is the remainder when $1! + 2! + 3! + 4! + \ldots + 99! + 100!$ is divided by 18?

SOLUTION: *$18 = 3 \times 6$, so 18 evenly divides $n!$ for $n \geq 6$*

Thus the remainder when $1! + 2! + 3! + 4! + \ldots + 99! + 100!$ is divided by 18 is the same as the remainder when $1! + 2! + 3! + 4! + 5! = 1 + 2 + 6 + 24 + 120 = 153 = 18(8)+9$ is divided by 18. So the remainder is $\boxed{9}$.