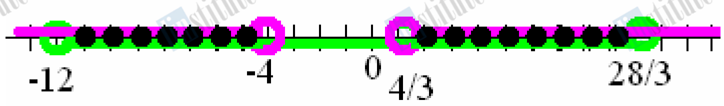
2004 Algebra II State Finals – Solutions

1. (C)

Since

$$8 < |3x+4| < 32 \Rightarrow 8 < |3x+4| \text{ and } |3x+4| < 32$$
$$8 < |3x+4| \Rightarrow 3x+4 > 8 \text{ or } 3x+4 < -8$$
$$3x > 4 \quad \text{or} \quad 3x < -12$$
$$x > 4/3 \quad \text{or} \quad x < -4$$
$$|3x+4| < 32 \Rightarrow 3x+4 < 32 \text{ and } 3x+4 > -32$$
$$3x < 28 \quad \text{and} \quad 3x > -36$$
$$x < 28/3 \quad \text{and} \quad x > -12$$

When these are plotted on a number line we see the following:



We see from the figure that there are 15 integral values in this set.

2. (B)

$$1 - \frac{4}{x} + \frac{4}{x^2} = 0 \Leftrightarrow x^2 - 4x + 4 = 0, x \neq 0 \Leftrightarrow (x-2)^2 = 0 \Rightarrow x = 2$$
$$\therefore \frac{2}{x} = \frac{2}{2} = 1$$

3. (D)

$$x = \frac{1}{y} \Rightarrow xy = 1, so \left(x - 1/x\right)\left(y + 1/y\right) = xy + x/y - y/x - 1/xy$$
$$= 1 + \frac{x^2 - y^2}{xy} - 1 = x^2 - y^2$$

4. (E) If a = 0, then $|b-0| = |b| - |0| = |b|$. If a = b, then $|b-a| = 0 = |b| - |a|$.

5. (A) All three points satisfy the equation $y = ax^2 + bx + c$, so

$$-3 = x, 6 = a + b - 3, 9 = 4a + 2b - 3 \Rightarrow$$
$$a + b = 9, 2a + b = 6 \Rightarrow a = -3, b = 12.$$
$$\therefore abc = (-3)(12)(-3) = 108$$

6. (B)

$$x \vartriangle y = x + xy + y \Rightarrow 8 \vartriangle z = 8 + 8z + z = 8 + 9z = 3$$
$$\Rightarrow 9z = -5 \Rightarrow z = -5/9$$

7. (A) The sum of the first for numbers is 1044. If x = 1 (the smallest possible value), then the average is 252.4, thus 218 is impossible. Note that the others are all possible since 1044 has a remainder of 4 when divided by 5, so 218x must have a remainder of 1 when divided by 5. This can only happen if x has a remainder of 2 when divided by 5. So when x = 2, 12, 37 and 57, we get the other 4 choices.

8. (D) Rate is distance divided by time so, Mr. Goebel's new rate is $\frac{2y}{s+10}$.

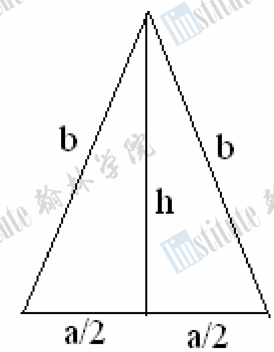
9. (B) The equation is $V(x)=V_0 r^x$, so $20=8\cdot r^{10} \Rightarrow r^{10}=2.5 \Rightarrow r=\sqrt[10]{2.5}$. So now $V(x)=8\left(\sqrt[10]{2.5}\right)^x$ and we want to find the value for x for which this equals 2000.

$$2000=8\left(\sqrt[10]{2.5}\right)^x \Rightarrow x=\frac{\log\left(2000/8\right)}{\log\left(\sqrt[10]{2.5}\right)}\approx 60.2588$$. The closest choice is 60.

10. (A) The area is

$$\tfrac{1}{2}\,base\times height=\tfrac{1}{2}a\cdot\sqrt{b^2-\left(a/2\right)^2}$$

$$=\frac{a}{2}\sqrt{\frac{4b^2-a^2}{4}}=\frac{a}{4}\sqrt{4b^2-a^2}$$



11. (E)

$x=10a+b, a\neq 0, b\neq 0$

$y=10b+a \Rightarrow x+y=11a+11b=11(a+b)$,

so the sum must be divisible by 11.

12. (E) Since x = 3, $y=2x+1 \Rightarrow y=2(3)+1=7$. Thus $7=m(3)+3 \Rightarrow 3m=4 \Rightarrow m=4/3$

13. (A) $(A+B)(A-B)=A^2-AB+BA-B^2$, which normally would simplify to A^2-B^2, but since matrix multiplications is not commutative, $-AB+BA\neq 0$.

14. (D) $\frac{3-i}{1+i}=\frac{(3-i)(1-i)}{(1+i)(1-i)}=\frac{3-3i-1i-1}{2}=\frac{2-4i}{2}=1-2i$.

15. (B) On the kth day the mass is M, on the $k-1^{th}$ day the mass is M/2, the $k-2^{nd}$ day the mass is M/4, and on the $k-3^{rd}$ day the mass is M/8.

16. (D)

$|x^2-5x|=6 \Rightarrow x^2-5x=6 \ \ or \ \ x^2-5x=-6$

$\therefore x^2-5x-6=0 \ \ or \ \ x^2-5x+6=0$

$\Rightarrow (x-6)(x+1)=0 \ \ or \ \ (x-2)(x-3)=0$

$\Rightarrow x=6,-1,2,3$

So there are 4 solutions whose sum is 10.

17. (D) The foot of the 25-foot ladder is 20 feet from the wall and the top of the ladder is 15 feet from the ground. (This is a 3-4-5 right triangle.) When the top slips down 8 feet, the base will be $\sqrt{25^2-7^2}=\sqrt{625-49}=\sqrt{576}=24$ feet from the wall, so it has moved 4 feet.

18. (D) There are five places around the 2 0 0 2 where one of the 1's could be placed. There are ${}_5C_2=\frac{5!}{3!2!}=10$ ways to place these. In addition, both of the 1's could be placed in each of the five locations, so there are a total of 15 different 6-digit numbers.

19. (C) The area $A=l\times w$, and the perimeter $2(l+w)=100$, so $l+w=50$ and $(l+w)^2=50^2\Rightarrow l^2+2lw+w^2=2500$. This last expression can be rewritten as $l^2+w^2+2lw=d^2+2A=2500\Rightarrow A=\frac{2500-d^2}{2}$.

20. (B) Let the two consecutive odd integers be $2n-1$ *and* $2n+1$. The difference then is $(2n+1)^2-(2n-1)^2=8n=128\Rightarrow n=16$. The product is $(2n+1)(2n-1)=4n^2-1=4(16^2)-1=1023$.

21. (C) Let the number of elements is S and T be m and n respectively. We know that the number of subsets of a set with m elements is 2^m, so we have $m-n=2$ and $2^m-2^n=96$. Let $m=n+2$ so that

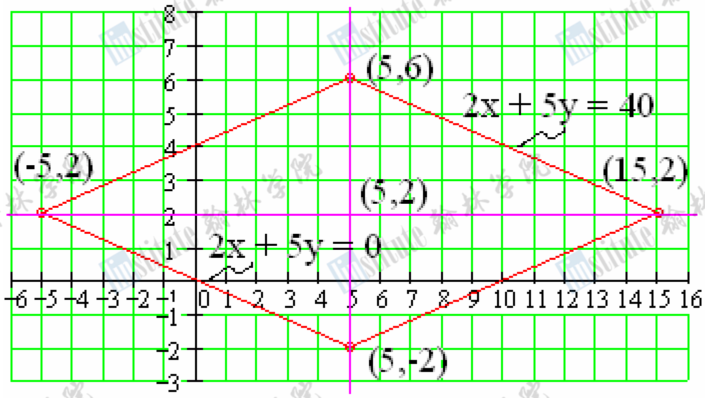
$$2^m-2^n=2^{n+2}-2^n=2^n\left(2^2-1\right)=96\Rightarrow 2^n\cdot 3=96$$
$$\Rightarrow 2^n=32\Rightarrow n=5\Rightarrow m=7$$

22. (D) If (-1,5) is a point on the graph of y = f(x) then the point (2,5) will be the corresponding point on the graph of $y=f(x-3)$, since this graph is the result of sliding the original graph to the right 3 units.

23. (C) First note that I is false since the product of two odd functions, like $f(x)=x$ *and* $g(x)=x^3$ is clearly an even function. Next note that the next two are clearly true. The final option, that $F(x)=\frac{f(x)+f(-x)}{2}$ is always even bears checking. First, if *f(x)* is even, then $F(x)=\frac{f(x)+f(-x)}{2}=\frac{f(x)+f(x)}{2}=f(x)$, which is even. If *f(x)* is odd, then $F(x)=\frac{f(x)+f(-x)}{2}=\frac{f(x)-f(x)}{2}=0$, which is a constant function, hence it is also even.

24. (D) We have $V=k\cdot h\cdot g^2$ and know that $216=k\cdot 30\cdot 1.5^2=67.5k\Rightarrow k=3.2$. Thus $960=3.2\cdot h\cdot 2^2\Rightarrow h=960\div\left(3.2\cdot 2^2\right)\Rightarrow h=75$.

25. (A) The graph of the solution set for this equation is made up of four line segments. If we think of the point (5,2) as the "center" of this graph, then the four cases are the "quadrants" about this center point. For example, if $x \ge 5, y \ge 2$, then both quantities inside the absolute values are positive, so the equation becomes $(2x-10)+(5y-10)=20 \Leftrightarrow 2x+5y=40$. The endpoints of this segment in the "quadrant" in question are (5,6) and (15,2). Similarly, if $x \le 5, y \le 2$, (the third "quadrant"), then $-(2x-10)+-(5y-10)=20 \Leftrightarrow -2x+-5y=0 \Leftrightarrow 2x+5y=0$, and the endpoints of this edge of the figure are (-5,2) and (5,-2).



The figure that is formed is a parallelogram and its area is $A=\frac{1}{2}d_1d_2$, where d_1 *and* d_2 are the lengths of the diagonals. So $A=\frac{1}{2}(20)(8)=80$.

26. (A) We know that $a_n+a_{n+1}+a_{n+2}=7$, so $a_{n+2}=7-(a_n+a_{n+1})$. Using this relation, we can generate the next several terms. They are:

$$a_6=7-(a_4+a_5)=7-(-6+8)=5$$
$$a_7=7-(a_5+a_6)=7-(8+5)=-6$$
$$a_8=7-(a_6+a_7)=7-(5+-6)=8$$

As you can see, the values will continue to cycle in groups of three. Since 2001 is divisible by 3, $a_{2001}=a_3=a_6=5$.

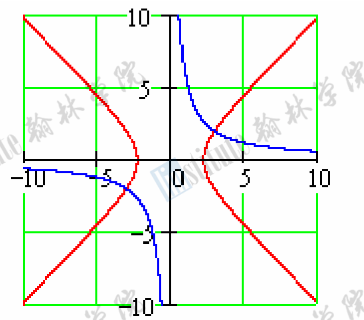
27. (B) Here we are decomposing a rational expression into partial fractions.

$\frac{N}{x-5}+\frac{3}{x+4}=\frac{N(x+4)+3(x-5)}{(x-5)(x+4)}=\frac{10x+13}{x^2-x-20}$, so for these last two fractions to be equal, the numerators must be equal, so we have $N(x+4)+3(x-5)=10x+13 \Leftrightarrow (N+3)x+(4N-15)=10x+13$.
From this last equation we see that $N+3=10,\ 4N-15=13 \Rightarrow N=7$.

28. (C) First notice that $15x+55y=2000 \Leftrightarrow 3x+11y=400$. Since 400 has remainder 1 when divided by 3, and 3x is always divisible by 3, we need a multiple of 11 what has remainder of 1 when divided by 3. This happens when y = 2, among others. So we have $3x+11(2)=400 \Rightarrow 3x=400-22=378 \Rightarrow x=126$. So the ordered pair (126,2) satisfies this equation. Each time we increase the y-values by 3, we

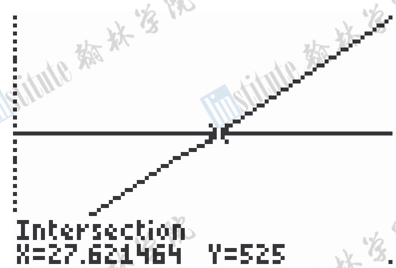
must decrease the x-values by 11, so the set of ordered pairs $\{(126-11k, 2+3k), 0 \le k \le 11\}$ gives us all ordered pairs of positive integers which satisfy the equation. The maximum sum of the coordinates occurs with the ordered pair (126,2) and is 128.

29. (C) The beauty of complex numbers is that there will always be two solutions to any quadratic equation (if we count double solutions as two solutions), so there will be two solutions. Checking to make sure we do not have a double solution, we see that $(a+bi)^2 = (a^2-b^2)+2abi = 5+12i$, so $a^2-b^2=5$ *and* $ab=6$. We could solve this system, or we could sketch graphs to see that there are two distinct solutions. Note that a is plotted on the horizontal axis.

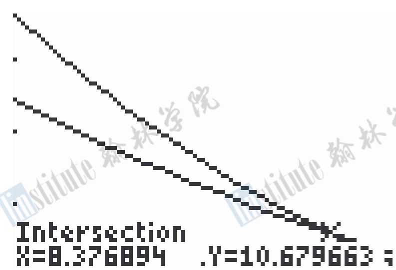


30. (A) $f(x)=2x^2+bx+10=2\left(x^2+\frac{b}{2}x+\frac{b^2}{4}\right)+\left(10-\frac{b^2}{2}\right)=2\left(x+\frac{b}{2}\right)^2+\left(10-\frac{b^2}{2}\right)$, so when b = 0 the vertex is (0,10) and as b increases, the vertex moves to the left at a rate of $\frac{b}{2}$ while it also moves down at a rate of $\frac{b^2}{2}$, which, when $b>1$ is a faster rate than the $\frac{b}{2}$.

31. (B) We are trying to find out when $1050-\sqrt[3]{9x^5+5x+503}=525$. The only way you have to do this is by using your calculator. Let $f(x)=\sqrt[3]{9x^5+5x+503}$ and $g(x)=525$ and graph these on a window that shows the intersection. Since all of the choices are around 27, a window of [26,29] would work. My TI-83 gives the figure to the right. Convert the decimal part of this answer to days by multiplying by 31 to get 27 months and 19 days.



32. (E) This is another calculator problem. We have $Y_1=50(0.822)^x$ and $Y_2=30(0.884)^x$. Both of these take x in years and return the current value in 1000's of dollars. So we want to know when $Y_2-Y_1=1$ or $Y_2=Y_1+1$. Graph these on your calculator and find the solution, since there is no way to solve this algebraically. According to this graph, the SUV is worth \$1000 more than the luxury car after 8.37 years.



33. (A) We need the composition $y(s(x))=y(x-32)=2((x-32)+12)=2(x-20)$

34. (E) Using properties of logs, we have

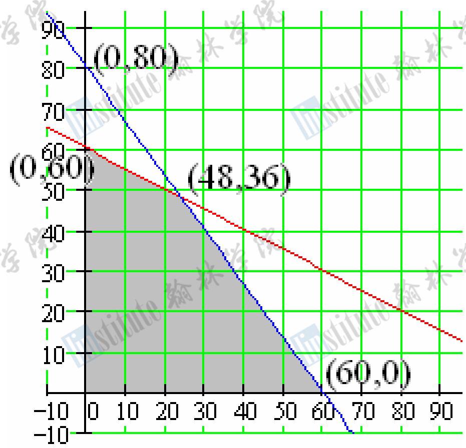
$\log_b 75 = \log_b\left(3\cdot 5^2\right) = \log_b 3 + 2\log_b 5 = 1.0986 + 2(1.6094) = 4.3174$

35. (B) $V = x^2h = 108 \Rightarrow h = \dfrac{108}{x^2}$. $SA = x^2 + 4xh = x^2 + 4x\left(\dfrac{108}{x^2}\right) = x^2 + \dfrac{4\cdot 108}{x}$. Without some Calculus, we are left to finding the minimum value numerically using our calculator. Graph this function on the window $[4 \le x \le 8]$ and use the "minimum" function to find the desired point. The minimum surface area occurs when x = 6 and the area is 108.

36. (E) This is a little linear programming problem. First we need to write and graph the constraint inequalities. If B stands for the number of Bookshelves and D the number of Desks, then we know that

$$5B + 10D \le 600;\ 4B + 3D \le 240;\ B \ge 0;\ D \ge 0.$$

The objective function, $P = 40B + 75D$ will have a maximum value at one of the corners of the region of feasible answers. The graph of this region and the coordinates of the corners are shown in the figure. The maximum profit is found by plugging the coordinates of each corner into the objective function. Doing this we get $P(0,60) = 4500$; $P(48,36) = 4620$; and $P(60,0) = 2400$.



37. (C) There are $(300)(400) = 120{,}000\ ft^2$ on the field. Since there are 144 square inches per square foot, there are $120{,}000\cdot 144 = 17{,}280{,}000\, in^2$ on the field. If, on the average, there are 3 ants per square inch, there will be $(17{,}280{,}000)\cdot 3 = 51{,}840{,}000$ and on the field. This is roughly 51 million ants.

38. (C) Since $2310 = 2\cdot 3\cdot 5\cdot 7\cdot 11$, there are five factors to choose from. To get only three numbers whose product is 2310, we would need combinations of either 3-1-1 of these factors or 2-2-1. There are ${}_5C_3\cdot 1\cdot 1 = \dfrac{5!}{3!2!} = 10$ and

${}_5C_2\cdot {}_3C_2\cdot 1 = \left(\dfrac{5!}{2!3!}\right)\left(\dfrac{3!}{2!1!}\right) = 10\cdot 3 = 30$. Thus there are 40 possible combinations.

39. (D) Since f is a linear function and $f(1) \le f(2)$, we know that the slope is greater than or equal to zero. Likewise, since $f(3) \ge f(4)$, the slope is less than or equal to

zero. Taken together, this means that the slope must be zero, making the function $f(x)=0x+5$, so $f(0)=5$.

40. (D) $\left(x^2-5x+5\right)^{x^2-9x+20}=1 \Rightarrow x^2-5x+5=1 \; or \; x^2-9x+20=0$. Solving the first we get $x^2-5x+5=1 \Leftrightarrow x^2-5x+4=0 \Leftrightarrow (x-4)(x-1)=0$, so x = 4 or 1. Solving the second we get $x^2-9x+20=0 \Leftrightarrow (x-4)(x-5)=0$, so x = 4 or 5. Thus the solutions are x = 1, 4, or 5.