

2004 State Mathematics Finals: Algebra I Solutions

1. D) $3x - c = 12 \Rightarrow 3x = 12 + c \Rightarrow x = \frac{12 + c}{3} = 4 + \frac{c}{3}.$

2. B) The function is a parabola with vertex (2,4), passing through (0,2). Thus $f(x) = a(x-h)^2 + k \Rightarrow 2 = a(0-2)^2 + 4 \Rightarrow -2 = 4(a) \Rightarrow a = -\frac{1}{2}$, so the function for this graph is $f(x) = -\frac{1}{2}(x-2)^2 + 4 = 4 - \frac{1}{2}(x-2)^2.$

3. C) $\frac{B(A-B) - A(A+B)}{A(A-B) + B(A+B)} = \frac{AB - B^2 - A^2 - AB}{A^2 - AB + AB + B^2} = \frac{-(A^2 + B^2)}{A^2 + B^2} = -1$

4. D) Let n and m be two natural numbers. Then

$$n \cdot m = 30 + n + m \Rightarrow n \cdot m - n = 30 + m \Rightarrow n(m-1) = 30 + m \Rightarrow n = \frac{30+m}{m-1}.$$

Making m as small as possible, $m = 2 \Rightarrow n = \frac{30+2}{2-1} = 32.$

5. A) This sequence "telescopes" if you multiply it out and rearrange the terms. It will look like:

$$\begin{aligned} & \left(\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{3} \right) + \left(\frac{1}{2} \cdot \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{5} \right) + \left(\frac{1}{2} \cdot \frac{1}{5} - \frac{1}{2} \cdot \frac{1}{7} \right) + \dots + \left(\frac{1}{2} \cdot \frac{1}{49} - \frac{1}{2} \cdot \frac{1}{51} \right) + \left(\frac{1}{2} \cdot \frac{1}{51} - \frac{1}{2} \cdot \frac{1}{53} \right) = \\ & \frac{1}{2} + \left(-\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} \right) + \left(-\frac{1}{2} \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{1}{5} \right) + \left(-\frac{1}{2} \cdot \frac{1}{7} + \frac{1}{2} \cdot \frac{1}{7} \right) + \dots + \left(-\frac{1}{2} \cdot \frac{1}{51} + \frac{1}{2} \cdot \frac{1}{51} \right) - \frac{1}{2} \cdot \frac{1}{53} = \\ & \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{53} = \frac{1}{2} \left(1 - \frac{1}{53} \right) = \frac{1}{2} \left(\frac{52}{53} \right) = \frac{26}{53} \end{aligned}$$

6. B) If the two quadratic expressions are to be the same for all x , then it follows that $x^2 - 49 + x^2 - 14x + 49 - 2px + 14p = kx^2 - 14kx + 49k$, thus $k = 2$ and $49k = 14p \Rightarrow p = 7.$

7. B) The paths of the cars can be shown as two right triangles, each with legs 3 and 7, making $2x$ the distance between the cars, $x^2 = 3^2 + 7^2 \Rightarrow x^2 = 58 \Rightarrow x = \sqrt{58}$, so $2x = 2\sqrt{58}.$

8. C) Note that to get the next term in the sequence, you need to add the next power of two to the previous answer. So $1 = 0 + 2^0, 3 = 1 + 2^1, 7 = 3 + 2^2, 15 = 7 + 2^3$, so the next term is $15 + 2^4 = 31.$

9. D) $36 = 6^2, 100 = 10^2, 225 = 15^2$, while $25 = 5^2, 49 = 7^2, 169 = 13^2$, so it appears as if Tina is selecting the squares of composite numbers while rejecting the squares of prime numbers. Since $64 = 8^2, 81 = 9^2, 196 = 14^2$, all are squares of composites, so she should keep them all.

10. B) If $x = 0$, then
 $4x^2 + 4y^2 - 16y = 9 \Rightarrow 4y^2 - 16y = 9 \Rightarrow 4y^2 - 16y - 9 = 0 \Rightarrow (2y - 9)(2y + 1) = 0$, so
 $y = 4.5$ or -0.5 . The distance between the points $(0, 4.5)$ and $(0, -0.5)$ is 5.

11. D) On a single flip of the coin the $P(H) = 0.6$, so $P(T) = 1 - 0.6 = 0.4$. There are three ways to flip exactly two heads, HHT, HTH, and THH. Each has $P(2H) = (.6)(.6)(.4)$, so the desired probability is $3(0.6)(0.6)(0.4) = 0.432$.

12. D) $\sqrt{x} = \sqrt[3]{2} \Rightarrow (\sqrt{x})^6 = (\sqrt[3]{2})^6 \Rightarrow x^3 = 2^2 \Rightarrow x^3 = 4$.

13. C) $x - y = c$ and $y = x^2 - 5 \Rightarrow x - (x^2 - 5) = c \Rightarrow x^2 - x + (c - 5) = 0$. When the discriminant is zero, there is exactly one solution, so
 $(-1)^2 - 4(1)(c - 5) = 0 \Rightarrow 1 - 4c + 20 = 0 \Rightarrow 4c = 21 \Rightarrow c = 5.25$.

14. A) $\frac{a}{3 - \sqrt{7}} = \frac{a}{3 - \sqrt{7}} \left(\frac{3 + \sqrt{7}}{3 + \sqrt{7}} \right) = \frac{a(3 + \sqrt{7})}{9 - 7} = \frac{a(3 + \sqrt{7})}{2}$.

15. A) When two lines do not intersect, they are parallel and have the same slope. Since $3x - 5y = 2 \Rightarrow 5y = 3x - 2 \Rightarrow y = \left(\frac{3}{5}\right)x - \frac{2}{5}$, so the slope is $\frac{3}{5} = 0.6$.

16. B) $x = y^2 + y + 1 \Rightarrow x = \left(y^2 + y + \frac{1}{4}\right) + 1 - \frac{1}{4} \Rightarrow x = \left(y + \frac{1}{2}\right)^2 + \frac{3}{4}$. Thus the vertex of this parabola is $\left(\frac{3}{4}, -\frac{1}{2}\right)$.

17. C) Suppose d = distance between cities A and B. Since rate $\times$ time = distance, or
 $\text{time} = \frac{\text{distance}}{\text{rate}}$, then the time from city A to B is $\frac{d}{60}$ and the time from B to A is $\frac{d}{40}$.
Then $\frac{d}{60} + \frac{d}{40} = \frac{2d}{ave} \Rightarrow \frac{40d + 60d}{60 \cdot 40} = \frac{2d}{ave} \Rightarrow ave = \frac{2 \cdot 60 \cdot 40}{40 + 60} = \frac{4800}{100} = 48$.

18. C) $x = 3y - 1 \Rightarrow x + 1 = 3y \Rightarrow y = \frac{x + 1}{3}$.

19. E) $x - \frac{a^2}{x+2} = 0 \Rightarrow x(x+2) - a^2 = 0 \Rightarrow x^2 + 2x - a^2 = 0$. Using the quadratic formula,

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-a^2)}}{2} = \frac{-2 \pm \sqrt{4 + 4a^2}}{2} = \frac{-2 \pm 2\sqrt{1 + a^2}}{2} = -1 \pm \sqrt{1 + a^2}.$$

20. D) Let x = rate to wash one car. Then $2 \cdot 6x = 15 \Rightarrow 12x = 15 \Rightarrow x = \frac{15}{12} = 1.25$. So 9 people can wash $9 \cdot 4 \cdot (1.25) = 45$ cars in the same amount of time.

21. E) Let s be the length of the sides of the square. Then $2s^2 = a^2 \Rightarrow s^2 = \frac{a^2}{2} \Rightarrow s = \frac{a}{\sqrt{2}}$
 and the perimeter is $4s = 4\left(\frac{a}{\sqrt{2}}\right) = 4\left(\frac{a}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{4a\sqrt{2}}{2} = 2a\sqrt{2}$.

22. A) $1 + 2 + 4 + \cdots + 2^{17} = 2^{18} - 1 = 262143$ pennies = \$2621.43 and $\$2621.43 - 1800 = \821.43 .

23. D) $\frac{1}{10} + \frac{1}{20} + \frac{1}{40} + \frac{1}{80} + \cdots = \frac{1}{5}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right)$, but since $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = c$,
 it follows that $\frac{1}{10} + \frac{1}{20} + \frac{1}{40} + \frac{1}{80} + \cdots = \frac{1}{5}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right) = \frac{1}{5}c = \frac{c}{5}$.

24. A) Let M = Mary's age in 1993. Since Mary is twice as old as Ben, his age in 1993 is $\frac{M}{2}$. In 2000 Mary is 7 years older than Ben, so
 $M + 7 = \left(\frac{M}{2} + 7\right) + 7 \Rightarrow \frac{M}{2} = 7 \Rightarrow M = 14$. Thus Mary was 14 in 1993, Ben was 7, and in 2004 Ben was $7 + 11 = 18$.

25. D) $(\sqrt{x} - \sqrt{3})(\sqrt{x} + \sqrt{3}) - (\sqrt{x} - \sqrt{2})^2 = x - 3 - (x - 2\sqrt{2x} + 2) = 2\sqrt{2x} - 5$.

26. A) Let x be the robber's time. Then $x - \frac{1}{4}$ = the sheriff's time. Using rate $\times$ time = distance, $16x = 20\left(x - \frac{1}{4}\right) \Rightarrow 16x = 20x - 5 \Rightarrow 4x = 5 \Rightarrow x = \frac{5}{4} = 1.25$ hr and $x - \frac{1}{4} = 1$ hr = 60 min.

27. C) $f(x) = g(x) \Rightarrow 11 - 2x = x^2 - 4 \Rightarrow 0 = x^2 + 2x - 15 \Rightarrow 0 = (x + 5)(x - 3)$. Thus $x = -5$ or $x = 3$. So the points of intersection are $(-5, 21)$ and $(3, 5)$ and $(-5, 21)$ is farthest from the origin.

28. C) $x^2 - 8 > -2x \Rightarrow x^2 + 2x - 8 > 0 \Rightarrow (x-2)(x+4) > 0$. The values which make the expression on the left equal to zero, namely $x = 2$ and $x = -4$ divide the number line into 3 regions; namely $x < -4$, $-4 < x < 2$, $x > 2$. By checking values in each of these intervals, we see that when $x < -4$ or $x > 2$ the inequality holds.

29. D) $x^{-2} + (+b)x^{-1} + ab = 0 \Rightarrow (x^{-1} + a)(x^{-1} + b) = 0 \Rightarrow x^{-1} + b = 0$ or $x^{-1} + a = 0 \Rightarrow x = \frac{-1}{b}$ or $x = \frac{-1}{a}$.

30. E) It can be shown that a, c, and e are sometimes false by letting $f(x) = x^2 + 1$. It can be shown too that b is false by letting $f(x) = 0$.

31. D) $(\sqrt{5} - \sqrt{3})^2 = 5 - 2\sqrt{15} + 3 = 8 - 2\sqrt{15}$ which is irrational. While this is all we really need, it is probably worthwhile to show that the others are rational. First $\sqrt{1.777\ldots} = \sqrt{\frac{16}{9}} = \frac{4}{3}$. The second, $(9.\overline{123})^2$ is a repeating decimal, hence rational. To find the fraction for $9.\overline{123}$, let $N = 9.\overline{123} \Rightarrow 1000N = 9123.\overline{123} \Rightarrow 1000N - N = 9,123.\overline{123} - 9.\overline{123} \Rightarrow 999N = 9114$, so $N = \frac{9114}{999}$. While 1.5129 is a terminating decimal, there is no guarantee its square root is

rational, but $1.5129 = \frac{15129}{10000} = \frac{123^2}{100^2} \Rightarrow (1.5129)^{-1/2} = \left(\frac{123^2}{100^2}\right)^{-1/2} = \left(\frac{100^2}{123^2}\right)^{1/2} = \frac{100}{123}$.

Finally, $\frac{(\sqrt{2} + 1)^2}{3 + \sqrt{8}} = \frac{2 + 2\sqrt{2} + 1}{3 + 2\sqrt{2}} = \frac{3 + 2\sqrt{2}}{3 + 2\sqrt{2}} = 1$.

32. B) Since (0,0), (42,0) and (3,2) are points on the curve, $f(x) = a(x-0)(x-42)$, and $f(3) = 2 = a(3-0)(3-42) \Rightarrow 2 = -117a \Rightarrow a = -\frac{2}{117}$. The maximum height occurs at the parabola's vertex, which is midway between the zeros, or $x = 21$, so $f(21) = -\frac{2}{117}(21)(21-42) = \frac{2 \cdot 21^2}{117} \approx 7.538$.

33. B) Since $64 = 8^2$, then $64^x = 8^3 \Rightarrow (8^2)^x = 8^3 \Rightarrow 8^{2x} = 8^3 \Rightarrow 2x = 3 \Rightarrow x = \frac{3}{2}$.

34. D) $322 + 341 + 401 = 2114$.

35. C) The duckweed covers 5 cm^2 on June 1 and doubles on June 5, 9, 13, 17, ... 29. On June 29th the duckweed covers $5 \cdot 2^7 = 640\text{ cm}^2$. On June 30th, the weed will cover $5 \cdot 2^{7.25} \approx 761.09\text{ cm}^2$.

36. E) The lengths of the sides of the triangle can be represented by $4x$, $3x$, and 25. So $(4x)^2 + (3x)^2 = 25^2 \Rightarrow 16x^2 + 9x^2 = 25^2 \Rightarrow 25x^2 = 25^2 \Rightarrow x = 5$, so the lengths of the legs are 15 cm and 20 cm and the area is $\frac{1}{2} \cdot 15 \cdot 20 = 150\text{ cm}^2$.

37. B) $2 \oplus (3 \oplus 5) = 2 \oplus \left(\frac{3 \cdot 5}{3+5} \right) = 2 \oplus \frac{15}{8} = \frac{2 \cdot 15/8}{2 + 15/8} = \frac{30/8}{31/8} = \frac{30}{31}$

38. B) $3(x \oplus 1) = 2x \oplus 5 \Rightarrow 3\left(\frac{x \cdot 1}{x+1}\right) = \frac{2x \cdot 5}{2x+5} \Rightarrow (3x)(2x+5) = (10x)(x+1)$, so $6x^2 + 15x = 10x^2 + 10x \Rightarrow -4x^2 + 5x = 0 \Rightarrow x(-4x+5) = 0$, so $x = 0$, or $x = \frac{5}{4}$. However, $x = \frac{5}{4}$ is the only positive solution.

39. A) $2x + 3y = c \Rightarrow 3y = c - 2x \Rightarrow y = \frac{c-2x}{3}$, so

$$3x + 4\left(\frac{c-2x}{3}\right) = 1 \Rightarrow 9x + 4c - 8x = 3 \Rightarrow x = 3 - 4c.$$

40. A) $\frac{1}{4} \cdot 3 \cdot 10 \cdot 12 + \frac{1}{4} \cdot 3 \cdot 11 \cdot 12 + \dots + \frac{1}{4} \cdot 3 \cdot 16 \cdot 12 + \frac{1}{4} \cdot 3 \cdot 17 \cdot 12$, which, by factoring equals $\frac{1}{4} \cdot 3 \cdot 12(10+11+12+\dots+16+17) = \frac{1}{4} \cdot 3 \cdot 12(10+17) \cdot \frac{8}{2} = 3 \cdot 12 \cdot 27 = 972$.