

Part I: Multiple Choice (Answers)

- 1) **Solution:** Let x be the number of students who failed.

$$60x + 84(60 - x) = 80 \cdot 60$$

$$24x = 2400$$

Answer: (e)

$$x = 10$$

- 2) **Solution:** The sum of two sides must be greater than the third.

Answer: (c)

- 3) **Solution:** The slopes of the two graphs must be negative reciprocals.

Answer: (a)

- 4) **Solution:**

$$s = r\theta$$

$$108 = r\left(\frac{9}{2}\right)$$

$$r = 24 \text{ ft.}$$

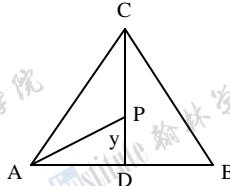
Answer: (d)

- 5) **Solution:**

$$y^2 + \left(\frac{x}{2}\right)^2 = (2y)^2$$

$$3y^2 = \left(\frac{x}{2}\right)^2$$

$$y = \frac{x}{2\sqrt{3}} = \frac{x\sqrt{3}}{6}$$



Answer: (a or c)

$$\overline{AP} = 2y = \frac{x\sqrt{3}}{3}$$

Note: The same answer was listed for a and c. Either correct answer was counted.

- 6) **Solution:** Let $x = \log_3 16$

$$y = \log_2 27$$

$$3^x = 16$$

$$2^y = 27$$

$$x = \frac{\ln 16}{\ln 3} = \frac{4 \ln 2}{\ln 3}$$

$$y = \frac{\ln 27}{\ln 2} = \frac{3 \ln 3}{\ln 2}$$

$$xy = \frac{4 \ln 2 \cdot 3 \ln 3}{\ln 3 \cdot \ln 2} = 12$$

Answer: (c)

7)

Solution:

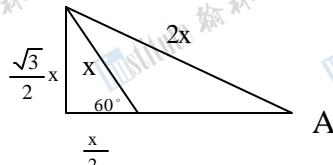
$$\begin{aligned}
 9 \tan\left(\frac{p}{4} - x\right) &= \tan\left(\frac{p}{4} + x\right) \\
 9 \left[\frac{\tan \frac{p}{4} - \tan x}{1 + \tan \frac{p}{4} \tan x} \right] &= \frac{\tan \frac{p}{4} + \tan x}{1 - \tan \frac{p}{4} \tan x} \\
 9 \left[\frac{1 - \tan x}{1 + \tan x} \right] &= \frac{1 + \tan x}{1 - \tan x} \\
 9(1 - \tan x)^2 &= (1 + \tan x)^2 \\
 9(1 - 2 \tan x + \tan^2 x) &= 1 + 2 \tan x + \tan^2 x \\
 8 \tan^2 x - 20 \tan x + 8 &= 0 \\
 (2 \tan x - 1)(\tan x - 2) &= 0 \\
 \tan x = 1/2; \quad \tan x = 2 & \\
 \text{Product of the roots} &= (\tan^{-1}(1/2))(\tan^{-1} 2) \approx .5133
 \end{aligned}$$

Answer: (b)

8)

Solution:

$$\begin{aligned}
 \sin A &= \frac{\frac{\sqrt{3}}{2}x}{2x} = \frac{\sqrt{3}}{4} \\
 A &= \sin^{-1} \frac{\sqrt{3}}{4} \approx 26^\circ
 \end{aligned}$$

**Answer:** (d)

9)

Solution:

$$\begin{aligned}
 f(2x+1) &= 4x^2 + 14x \Rightarrow f(x) = f\left(2\left(\frac{x-1}{2}\right) + 1\right) \\
 f(x) &= 4\left(\frac{x-1}{2}\right)^2 + 14\left(\frac{x-1}{2}\right) \\
 &= x^2 - 2x + 1 + 7x - 7 \\
 f(x) &= x^2 + 5x - 6 \text{ which has roots of } 1 \text{ and } -6.
 \end{aligned}$$

Thus the sum of the roots is -5 .**Answer:** (d)

10)

Solution:

$$\begin{aligned}
 x^2 + (\cot A \cos A)x - 1 &= 0 \quad \text{with } 0 < A < p/2 \\
 x^2 + (\cos^2 A / \sin A)x - 1 &= 0 \\
 (\sin A) \cdot x^2 + (\cos^2 A) \cdot x - \sin A &= 0 \\
 x &= \frac{-\cos^2 A \pm \sqrt{\cos^4 A + 4\sin^2 A}}{2\sin A} \\
 x &= \frac{-\cos^2 A \pm \sqrt{\cos^4 A + 4(1 - \cos^2 A)}}{2\sin A}
 \end{aligned}$$

$$x = \frac{-\cos^2 A \pm \sqrt{\cos^4 A - 4\cos^2 A + 1}}{2\sin A}$$

$$x = \frac{-\cos^2 A \pm \sqrt{(\cos^2 A - 2)^2}}{2\sin A}$$

$$x = \frac{-\cos^2 A \pm (\cos^2 A - 2)}{2\sin A}$$

$$x_1 = \frac{-\cos^2 A + (\cos^2 A - 2)}{2\sin A} = \frac{-2}{2\sin A} = -\csc A$$

$$x_2 = \frac{-\cos^2 A - (\cos^2 A - 2)}{2\sin A} = \frac{-2\cos^2 A + 2}{2\sin A} = \sin A$$

$$\text{Thus } x_1 + x_2 = \sin A - \csc A$$

Answer: (b) (A quicker answer may be found by using the sum of the roots.)

11) **Solution:**

$$\begin{aligned} e^2 x^{\ln x} &= x^3 \\ \ln(e^2 x^{\ln x}) &= \ln(x^3) \\ \ln e^2 + \ln x^{\ln x} &= 3 \ln x \\ 2 + \ln x \ln x &= 3 \ln x \\ (\ln x)^2 - 3 \ln x + 2 &= 0 \\ (\ln x - 2)(\ln x - 1) &= 0 \\ \ln x = 2 \Rightarrow x &= e^2 \\ \ln x = 1 \Rightarrow x &= e \end{aligned}$$

Answer: (c)

Thus the sum of the roots is $e^2 + e \approx 10.107$

12) **Solution:** The roots will be real and unequal when $b^2 - 4ac > 0$. For the equation above, we have $k^2 - 4 \cdot 1 \cdot 1$, and so k must be greater than 2. The probability of this happening is $2/3$

Answer: (b)

13) **Solution:**

$$\begin{array}{r} 2201_{\text{three}} \\ \times \underline{212}_{\text{three}} \\ \hline 12102 \\ 22010 \\ \hline 1210200 \\ \hline 2022012_{\text{three}} \end{array}$$

Answer: (d)

14) **Solution:** $(a * b) * c = (b + 1) * c = c + 1$
 $a * (b * c) = a * (c * 1) = c + 2$
Answer: (e) $c + 1$ is never equal to $c + 2$

15) **Solution:** The total number of ways to choose 7 students is $7! = 5040$. The number of ways to choose a girl or boy alternating is $4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 144$. Thus the quotient of the alternating ways divided by the total number of ways is $\frac{144}{5040} = \frac{1}{35}$.

Answer: (a)

16) **Solution:** Let x be the required time in hours.

$$\frac{x}{2} - \frac{x}{5} = \frac{1}{2}$$

Answer: (b) $x = \frac{5}{3}$ or 1 hour and 40 minutes

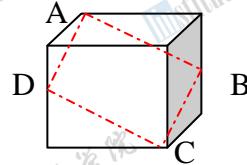
17) **Solution:** $\sqrt[3]{x + 3p + 1} = \sqrt[3]{x} + 1$ Let $h = \sqrt[3]{x}$
 $\sqrt[3]{x + 3p + 1} = h + 1$
 $x + 3p + 1 = h^3 + 3h^2 + 3h + 1$
 $h^3 + 3p + 1 = h^3 + 3h^2 + 3h + 1$
 $3h^2 + 3h - 3p = 0$
 $h^2 + h - p = 0$
 $b^2 - 4ac = 1 + 4p \geq 0$ or $p \geq -\frac{1}{4}$

Answer: (b)

18) **Solution:** $\tan\left(\tan^{-1}\frac{x}{10} + \tan^{-1}\frac{1}{x+1}\right) = \tan\frac{p}{4}$
 $\frac{\frac{x}{10} + \frac{1}{x+1}}{1 - \frac{x}{10} \cdot \frac{1}{x+1}} = 1$ or $\frac{x}{10} + \frac{1}{x+1} = 1 - \frac{x}{10} \cdot \frac{1}{x+1}$
 $x(x+1) + 10 = 10(x+1) - x$
 $x^2 + x + 10 = 10x + 10 - x$
 $x^2 - 8x = 0$
 $x = 0, 8$

Answer: (a)

- 19) **Solution:** ABCD is a rhombus. \overline{AC} has length $\sqrt{3} k$ and the diagonal \overline{BD} has length $\sqrt{2} k$.



$$\text{Area } ABCD = \frac{\sqrt{2}k \cdot \sqrt{3}k}{2} = \frac{\sqrt{6}k^2}{2}$$

Answer: (b)

- 20) **Solution:** Let x , $x + 1$, and $x + 2$ be three consecutive integers.

Case 1: If x is even, then $x + 2$ is also even, so 4 divides the product.

Case 2: If x is odd, write x as $2k - 1$. Two sub-cases exist.

2.1) If k is even, say $k = 2n$, then

$x + 1 = (2k - 1) + 1 = 2k = 2(2n)$. So 4 divides the product.

2.2) If k is odd, say $k = 2n + 1$, then

$x + 1 = (2k - 1) + 1 = 2k = 4n + 2 = 2(2n + 1)$.

So 4 does not divide the product.

Thus the probability is $1/2(1 + 1/2) = 3/4$.

Alternate Solution:

Any integer can be written in one of the six forms: $6k$, $6k + 1$, $6k + 2$, $6k + 3$, $6k + 4$, and $6k + 5$. Let $?$ denote the product of 3 consecutive integers and consider the six cases below.

?		Probability M is divided by 12
$6k(6k + 1)(6k + 2)$	$12k(6k + 1)(3k + 1)$	Always divisible by 12
$(6k + 1)(6k + 2)(6k + 3)$	$6(6k + 1)(3k + 1)(2k + 1)$	Divisible by 12 when k is odd
$(6k + 2)(6k + 3)(6k + 4)$	$12(3k + 1)(2k + 1)(3k + 2)$	Always divisible by 12
$(6k + 3)(6k + 4)(6k + 5)$	$6(2k + 1)(3k + 2)(6k + 5)$	Divisible by 12 when k is even
$(6k + 4)(6k + 5)(6k + 6)$	$12(3k + 2)(6k + 5)(k + 1)$	Always divisible by 12
$(6k + 5)(6k + 6)(6k + 7)$	$6(6k + 5)(k + 1)(6k + 7)$	Divisible by 12 when k is odd

The probability of each case is $\frac{1}{6}$. Thus $\frac{1}{6}\left(1 + \frac{1}{2} + 1 + \frac{1}{2} + 1 + \frac{1}{2}\right) = \frac{3}{4}$ is the probability that $?$ will be divisible by 12.

Answer: (e)

Part II: Integer Answer (15 Problems)

1) **Solution:** $2^{10} - 2^7 = 1024 - 128 = 896$

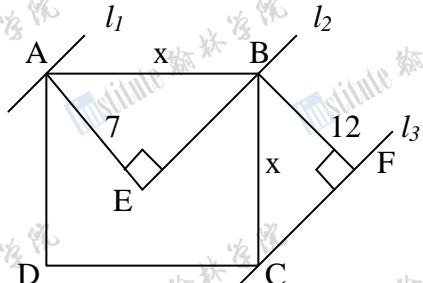
Answer: 896

2) **Solution:** $\triangle ABE \cong \triangle BCF$

$$\frac{7}{x} = \frac{\sqrt{x^2 - 12^2}}{x}$$

$$49 = x^2 - 12^2$$

$$x^2 = 193$$



Answer: 193

3) **Solution:**

$$r_{\text{avg}} = \frac{2d}{t_1 + t_2} = \frac{2d}{\frac{d}{v+40} + \frac{d}{v-40}} = \frac{2}{\frac{2v}{v^2 - 40^2}} = \frac{v^2 - 40^2}{v}$$

$$393 = \frac{v^2 - 40^2}{v}$$

$$v^2 - 396v - 1600 = 0$$

$$(v - 400)(v + 4) = 0$$

$$v = 400$$

Answer: 400 mph

4) **Solution:**

$$\frac{2}{9!} + \frac{2}{7!3!} + \frac{1}{5!5!} = \frac{20}{10!} + \frac{240}{10!} + \frac{252}{10!} = \frac{512}{10!} = \frac{2^9}{10!}$$

$a = 9$ and $b = 10$ Thus $a + b = 19$

Answer: 19

5) **Solution:**

Note that $\frac{1}{\log_2 100!} = \frac{\ln 2}{\ln 100!}$, $\frac{1}{\log_3 100!} = \frac{\ln 3}{\ln 100!}$, etc

$$\begin{aligned} \text{Thus } A &= 100 \left(\frac{\ln 2}{\ln 100!} + \frac{\ln 3}{\ln 100!} + \dots + \frac{\ln 100}{\ln 100!} \right) \\ &= 100 \left(\frac{\ln (2 \cdot 3 \cdot 4 \cdots 100)}{\ln 100!} \right) = 100 \cdot 1 = 100 \end{aligned}$$

Answer: 100

6) **Solution:** Let $\frac{1}{n} = .abcdef\overline{abcdef}$

$$\text{Then } 10^6 \left(\frac{1}{n} \right) = \text{abcdef}.\overline{\text{abcdef}}$$

$$999999 \left(\frac{1}{n} \right) = \text{abcdef}$$

$$999999 = n(\text{abcdef}) \text{ Now consider the factors of 999999}$$

$999999 = 3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$ The factors that satisfy the given conditions are 7, 13, 21, 39, 63, & 77 and their sum is 220.

Answer: 220

7) **Solution:** Since $1 + \sqrt{2}$ is a root, $1 - \sqrt{2}$ must also be a root.

$$(1 + \sqrt{2})^4 + a(1 + \sqrt{2}) + b = 0$$

$$(1 - \sqrt{2})^4 + a(1 - \sqrt{2}) + b = 0$$

$$(1 + \sqrt{2})^4 - (1 - \sqrt{2})^4 + 2\sqrt{2} a = 0$$

$$8\sqrt{2} + 16\sqrt{2} + 2\sqrt{2} a = 0$$

$$4 + 8 + a = 0$$

$$a = -12, b = -5$$

$$\text{Thus } b - a = 7$$

Answer: 7

8) **Solution:** Assume that side a is opposite $\angle A$.

$$(a + b)(a - b) = c(b + c) \quad \text{or} \quad a^2 - b^2 = c^2 + bc$$

$$a^2 = b^2 + c^2 + bc = b^2 + c^2 - 2bc \cos A \quad (\text{By Law of Cosines})$$

$$bc = -2bc \cos A$$

$$\cos A = -1/2 \quad \text{or} \quad \angle A = 120^\circ$$

Answer: 120°

9) **Solution:** Let N^2 be a perfect square in the sequence. Then $N^2 - 1$ must be a multiple of 7. Thus $(N+1)(N-1) = 7m$. So either $(N+1)$ or $(N-1)$ is a multiple of 7.

Perfect Square	1 st	2 nd	3 rd	4 th	5 th	6 th	...	100 th
m	0	1	1	2	2	3		50
N + 1		7		14		21		350
N - 1	0		7		14			
N	1	6	8	13	15	20		349
N^2	1	36	64	169	225	400		349^2

Answer: $349^2 = 121801$

10) **Solution:** $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} = \frac{a+b+c}{abc} = \frac{\text{sum of the roots}}{\text{product of the roots}} = \frac{4/k}{1/k} = 4.$

Answer: 4

11) **Solution:** $f(-1) = f(0) + 1 = k + 1$
 $f(-2) = k + 1 + 4$
 $f(-3) = k + 1 + 4 + 7$
 \vdots
 $f(-50) = k + 1 + 4 + 7 + \dots + 148 = 4000$
 $k + \frac{(1+148)}{2} \cdot 50 = 4000 \quad \text{or} \quad k + 3725 = 4000$
 $k = 275$

Answer: 275

12) **Solution:**
$$\begin{aligned}(1+x+x^2)^{12} &= [1+(x+x^2)]^{12} \\&= 1^{12} + 12 \cdot 1^{11}(x+x^2) + 66 \cdot 1^{10}(x+x^2)^2 + 210 \cdot 1^9(x+x^2)^3 + \dots \\&\quad (\text{no terms involving } x^3) \\&= 66 \cdot 2x^3 + 220x^3 = 352x^3\end{aligned}$$

Answer: 352

13) **Solution:** Note that there is an intersection at $x = 0$. There can be no intersection beyond $x = 200$ which is approximately 64π . Within each period $(0, 2\pi)$, $(2\pi, 4\pi)$, ..., $(62\pi, 64\pi)$ there must be 2 intersections. These 64 intersections plus the one at zero yields 65 intersections.

Answer: 65

14) **Solution:** Let the 5 points be ABCDE. Note that to paint the 5 points in the prescribed fashion, 2 colors must be used twice and one color just once. Also, note that if point A is a fixed color, then there are only two ways to paint the star. Consider one such set of colors, where red is the single color used. Then if A is painted red there are only two ways to paint the rest of the star. A similar situation occurs if the points B, C, D, or E are painted with the red. For this set of colors there are 10 ways to paint the star. Since there are three different sets of colors, there are a total of 30 ways to paint the star.

Answer: 30

15) **Solution:** Note that $7^4 = 2401 = 2400 + 1 = 24(100) + 1$
Now any number of the form $(100t + 1)^{10}$ is a number of the form $1000t + 1$, so 7^{40} is of that form.

$$7^{1166} = 7^{1160+6} = 7^{1160} \cdot 7^6 = (7^{40})^{29} \cdot 7^6$$

Which has the same last 3 digits as 7^6 or 649.

Answer: 649

