Algebra II
State Mathematics Finals
Solutions

1. B. $(2x-3)^2 - m = 0 \Rightarrow 4x^2 - 12x + 9 - m = 0 \Rightarrow x^2 - 3x + \frac{9-m}{4} = 0$, so the sum of the roots is –(-3) = 3.

2. C. $\sqrt[x]{\sqrt{\sqrt[3]{25}}} = 25^{\frac{1}{6x}} = \left(5^2\right)^{\frac{1}{6x}} = 5^{\frac{1}{3x}}$

3. B. $\frac{1}{x+1} > \frac{1}{x-2} \Rightarrow \frac{x-2}{(x+1)(x-2)} - \frac{x+1}{(x+1)(x-2)} > 0 \Rightarrow$
$\frac{-3}{(x+1)(x-2)} > 0 \Rightarrow -1 < x < 2$

4. A. $9^{x+1} + 9^{x+2} + 9^{x+3} + 9^{x+4} + 9^{x+5} = 22143 \Rightarrow 9^x\left(9 + 9^2 + 9^3 + 9^4 + 9^5\right) = 22143$
$\Rightarrow \left(3^2\right)^x (66429) = 22143 \Rightarrow 3^{2x} = \frac{1}{3} \Rightarrow 3^{2x} = 3^{-1} \Rightarrow 2x = -1 \Rightarrow x = -\frac{1}{2}$

5. D. $\sqrt{1+x+\sqrt{x}} = \sqrt{x+\sqrt{x+7}} \Rightarrow 1+x+\sqrt{x} = x+\sqrt{x+7} \Rightarrow$
$1+\sqrt{x} = \sqrt{x+7} \Rightarrow \left(1+\sqrt{x}\right)^2 = \left(\sqrt{x+7}\right)^2 \Rightarrow$
$1+2\sqrt{x}+x = x+7 \Rightarrow 2\sqrt{x} = 6 \Rightarrow \sqrt{x} = 3 \Rightarrow x = 9$

6. B. Let x be the number of blue marbles.
$\frac{x}{x+80+24} = \frac{1}{5} \Rightarrow 5x = x + 104 \Rightarrow x = 26$.

7. E. Since $f(x) = f(x-2) + x,\ f(7) = f(5) + 7 \Rightarrow 11 = f(5) + 7 \Rightarrow f(5) = 4$.

8. A. $x^4 - 6x^2 + 5 = \left(x^2-5\right)\left(x^2-1\right) = \left(x^2-5\right)(x+1)(x-1)$, so the sum of the factors is $\left(x^2-5\right) + (x+1) + (x-1) = x^2 + 2x - 5$.

9. D. $\frac{2x-11}{x^2-5x-14} = \frac{B}{x-7} + \frac{C}{x+2} \Rightarrow$
$\frac{2x-11}{(x-7)(x+2)} = \frac{B(x+2)}{(x-7)(x+2)} + \frac{C(x-7)}{(x-7)(x+2)} \Rightarrow$
$2x - 11 = B(x+2) + C(x-7) \Rightarrow$
$2x - 11 = (B+C)x + (2B-7C) \Rightarrow B + C = 2$

10. C. To find the inverse, exchange x and y in the equation $y=\frac{x+2}{x-3}$ and solve for y. Solving for y,

$$xy-3x=y+2 \Rightarrow xy-y=3x+2 \Rightarrow y(x-1)=3x+3 \Rightarrow$$
$$y=\frac{3x+2}{x-1}, x \neq 1 \text{ or } f^{-1}(x)=\frac{3x+2}{x-1}, x \neq 1.$$

11. B.
$$\frac{x^3-y^3}{x^4+x^2y^2+y^4}=\frac{(x-y)(x^2+xy+y^2)}{x^4+x^2y^2+y^4}=$$
$$\frac{(x-y)(x^2+xy+y^2)}{(x^2+xy+y^2)(x^2-xy+y^2)}=\frac{(x-y)}{(x^2-xy+y^2)}$$

Using long division one can show that x^2+xy+y^2 is a factor of $x^4+x^2y^2+y^4$.

12. E. This function has a vertical asymptote at $x=0$, but of more importance to this question is the horizontal asymptote at $y=1$, since $F(x)=\frac{x+3}{x}=1+\frac{3}{x}$. Since y can never equal 1, but can equal anything else, the range of this function is $\{x|x \in \text{Reals}, x \neq 1\}$.

13. C.
$$\frac{9+3^{2x}}{10}=3^x \Rightarrow 3^2+3^{2x}=10\cdot 3^x \Rightarrow 3^{2x}-10\cdot 3^x+9=0 \Rightarrow$$
$$(3^x-9)(3^x-1)=0 \Rightarrow 3^x-9=0 \text{ or } 3^x-1=0 \Rightarrow$$
$$3^x=9 \text{ or } 3^x=1 \Rightarrow x=2 \text{ or } x=1$$

If $x=2,\ x^2+x+1=7$ and if $x=0,\ x^2+x+1=0$, so there are 2 solutions, 1 or 7.

14. E. If $y^2=|5-4y| \Rightarrow y^2=5-4y \text{ or } y^2=-(5-4y) \Rightarrow$
$y^2+4y-5=0 \text{ or } y^2-4y+5=0.$

The first of these, $y^2+4y-5=0 \Rightarrow (y+5)(y-1)=0$ has solutions -5 and 1 with a sum of 4. The second equation has no real solutions.

15. D. $x>p$ and $y>q \Rightarrow (x+y)>(p+y)>(p+q)$. Let $x=y=0$ and $p=q=-1$. Then I and iii are false. So ii only is true.

16. E. $({}_{10}C_2)({}_8C_5)({}_3C_3)=45\cdot 56\cdot 1=2520$.

17. A. For stacks of 8 with 7 bills left over, $x = 8n + 7$, for some integer $n, n \ge 1$, so $x \in \{15, 23, 31, 39, \cdots, 95\}$. Similarly, $x = 6k + 3$, for some integer $k, k \ge 1$, so $x \in \{9, 15, 21, \cdots, 97\}$, and $x = 7m + 4$ for some integer $m, m \ge 1$, so $x \in \{11, 18, 25, \cdots, 95\}$. Since 39 is the only element of all 3 sets, we seek to find the remainder when 39 is divided by 5. That remainder is 4.

18. C. $y = 2x^2$ and $y = x^2 + x + 6 \Rightarrow 2x^2 = x^2 + x + 6 \Rightarrow$.

$x^2 - x - 6 = 0 \Rightarrow (x-3)(x+2) = 0 \Rightarrow x = 3$ or $x = -2$

$\Rightarrow y = 2 \cdot 3^2 = 18$ or $y = 2(-2)^2 = 9$

The parabolas intersect at (3,18) and (-2,8). The line through (3,18) and (-2,8) has slope $m = 2$ and equation $2x - y + 12 = 0$.

19. E. $6x^2 + 11x + k = 0 \Rightarrow x^2 + \frac{11}{6}x + \frac{k}{6} = 0$, therefore $-\dfrac{11}{6}$ is the sum of the roots of the equation.

20. A. $f(i) = \left(\dfrac{i^4 - i^3 + i^2 - i + 1}{i}\right)^3 = \left(\dfrac{1 + i - 1 - i + 1}{i}\right)^3 = \left(\dfrac{1}{i}\right)^3 = \dfrac{1}{i^3} \cdot \dfrac{i}{i} = \dfrac{i}{1} = i$.

21. C.

$$-\log_2\left(x - \sqrt{x^2 - 1}\right) = \log_2\left(x - \sqrt{x^2 - 1}\right)^{-1} = \log_2\left(\frac{1}{x - \sqrt{x^2 - 1}}\right) =$$

$$\log_2\left(\frac{1}{x - \sqrt{x^2 - 1}} \cdot \frac{x + \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}}\right) = \log_2\left(\frac{x + \sqrt{x^2 - 1}}{x^2 - \left(x^2 - 1\right)}\right) = \log_2\left(x + \sqrt{x^2 - 1}\right)$$

22. A.

$$\ln \sqrt[a]{e^5} + \frac{2 \ln \sqrt[b]{e^4}}{a} - \frac{5 \ln \sqrt[a]{e^3}}{3} = 16 \Rightarrow \ln e^{\frac{5}{a}} + \tfrac{2}{a} \ln e^{\frac{4}{b}} - \tfrac{5}{3} \ln e^{\frac{3}{a}} = 16$$

$$\Rightarrow \frac{5}{a} + \frac{2}{a} \cdot \frac{4}{b} - \frac{5}{3} \cdot \frac{3}{a} = 16 \Rightarrow 5b + 9 - 5b = 16ab \Rightarrow ab = \frac{1}{2}$$

23. B.

$$8 \cdot {}_nP_3 = 2 \cdot {}_nP_4 \Rightarrow 4 \cdot {}_nP_3 = {}_nP_4 \Rightarrow 4n(n-1)(n-2) = n(n-1)(n-2)(n-3)$$

$$\Rightarrow 4 = n - 3 \Rightarrow n = 7$$

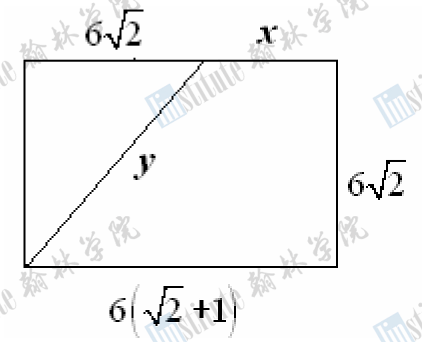
24. C. If $3w = \dfrac{x}{2}$, then $6w = x$, so $F(3w) = (6w)^2 + (6w) + 3 = 9 \Rightarrow 36w^2 + 6w - 6 = 0$

$$\Rightarrow 6w^2 + w - 1 = 0 \Rightarrow (3w - 1)(2w + 1) = 0 \Rightarrow w = \frac{1}{3} \text{ or } w = -\frac{1}{2}.$$

So the sum of these values is $-\dfrac{1}{6}$.

25. B. $F(2x+5)=F(5)\Rightarrow x=0$. When $x=0$, then $y=3F(5)-5=3\cdot 7-5=16$.
$x+y=0+16=16$.

26. D.



$$x=6\left(\sqrt{2}+1\right)-6\sqrt{2}=6$$
$$y=6\sqrt{2}\cdot\sqrt{2}=12$$
Perimeter $=6+6\sqrt{2}+6\left(\sqrt{2}+1\right)+12=$
$24+12\sqrt{2}$

27. D. $\log_5\sqrt{2}=\dfrac{\log\sqrt{2}}{\log 5}=x\Rightarrow\dfrac{\log 5}{\log\sqrt{2}}=\log_{\sqrt{2}}5=\dfrac{1}{x}$

28. E.
$$x^2-(1-2i)x=\left(\tfrac{1}{2}+i\right)\Rightarrow 2x^2+(-2+4i)x-1-2i=0$$
$$x=\frac{-(-2+4i)\pm\sqrt{(-2+4i)^2-4(2)(-1-2i)}}{4}=\frac{2-4i\pm\sqrt{-4}}{4}$$
$$x=\frac{2-4i\pm 2i}{4},x=\frac{1-i}{2}\text{ or }x=\frac{1-3i}{2}$$

29. C. $x^{\frac{1}{2}}\cdot x^{-\frac{1}{4}}\cdot x^{\frac{1}{8}}\cdot x^{-\frac{1}{16}}\cdots\cdot=x^{\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\cdots}$ The exponent in this last expression is an infinite geometric series with sum $\dfrac{1/2}{1-\left(-1/2\right)}=\dfrac{1/2}{3/2}=\dfrac{1}{3}$, so $x^{\frac{1}{2}}\cdot x^{-\frac{1}{4}}\cdot x^{\frac{1}{8}}\cdot x^{-\frac{1}{16}}\cdots\cdot=x^{\frac{1}{3}}=\sqrt[3]{x}$.

30. E. Using substitution with the first two equations and then the first and third equations results in the system of two equations in two variables
$$\begin{cases}kx+(-x+5)=17\\ x+k(-x+5)=-12\end{cases}\Rightarrow\begin{cases}kx-x=12\\ -kx+x=-2-5k\end{cases}\Rightarrow 0=10-5k\Rightarrow k=2\Rightarrow 2k+1=5$$

31. C. To find f^{-1}, $x=y+\dfrac{1}{4}\Rightarrow 4x=4y+1\Rightarrow y=\dfrac{4x-1}{4}\Rightarrow f^{-1}(x)=x-\dfrac{1}{4}$. Then
$$\left(h\circ f^{-1}\right)\left(\frac{1}{2}\right)=h\left(f^{-1}\left(\frac{1}{2}\right)\right)=h\left(\frac{1}{2}-\frac{1}{4}\right)=h\left(\frac{1}{4}\right)=\left(\frac{1}{4}\right)^{3/2}=\frac{1}{8}.$$

32. A. The 110[th] group has 110 numbers beginning with 5996. So the sum S is $S = \frac{110}{2}(5996 + 6105) = 665,555$.

33. C. $f(a) = b \Rightarrow a^2 = b$ and $f(c) = d \Rightarrow c^2 = d$. Also slope $m = \frac{d-b}{c-a} = \frac{c^2 - a^2}{c-a} = 5$. Since the x-coordinates of P and Q differ by 1, $c - a \neq 0$, so $m = c + a = 5$. Also $a + 1 = c$, or $c + 1 = a$. If $a + 1 = c$, then $c + a = 5 \Rightarrow 2a + 1 = 5 \Rightarrow a = 2$ and $c = 3$. If $c + 1 = a$, then $c + a = 5 \Rightarrow c = 2$ and $a = 3$. Then $b + d = a^2 + c^2 = 2^2 + 3^2 = 4 + 9 = 13$

34. D. $x^2 + y^2 - 10x + 6y + 9 = 0 \Rightarrow (x-5)^2 + (y+3)^2 = 25$. So L passes through the center (5,-3) and (7,1) and has slope $m = \frac{-3-1}{5-7} = 2$ and the equation of L is $y = 2x - 13$. The only choice that lies on the line is (8,3), choice D.

35. A. $x^4 + 2x^3 + 9x^2 + 18x = 0 \Rightarrow x(x+2)(x^2+9) = 0 \Rightarrow x = 0, -2,$ or $\pm 3i$. The sum of the squares of all solutions is $0^2 + (-2)^2 + (3i)^2 + (-3i)^2 = 4 - 9 - 9 = -14$.

36. D. $ax + by = (a-b)^2$ and $ax - by = a^2 - b^2 \Rightarrow 2ax = a^2 - 2ab + a^2$

$\Rightarrow ax = a^2 - ab \Rightarrow x = a - b$ and $y = b - a \Rightarrow x - y = a - b - (b - a)$

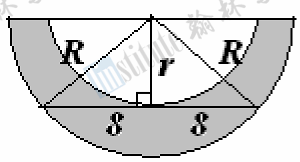
$= 2(a-b)$

37. D. By substitution,

$$7\cos x + 3\sin x = 7\sin x + 3\cos x \Rightarrow \sin x = \cos x \Rightarrow x = \pi/4$$

$$\Rightarrow y = 7\cos \pi/4 + 3\cos \pi/4 = 7 \cdot \sqrt{2}/2 + 3 \cdot \sqrt{2}/2 = 5\sqrt{2}$$

38. B.



Let r = radius of smaller circle and R = radius of the larger circle. Then $r^2 + 8^2 = R^2 \Rightarrow R^2 - r^2 = 8^2$. The area of the shaded region = Area of larger semicircle – area of smaller semicircle = $\frac{\pi}{2}R^2 - \frac{\pi}{2}r^2 = \frac{\pi}{2}(R^2 - r^2) = \frac{\pi}{2} \cdot 64 = 32\pi$

39. E. $\frac{(x^2 - x - 6)}{x-5} \geq 0 \Rightarrow \frac{(x-3)(x+2)}{(x-5)} \geq 0 \Rightarrow -2 \leq x \leq 3$ or $x > 5$.

40. D $a^2 - b^2 = c^2 - d^2 = 9^2 \Rightarrow (a-b)(a+b) = (c-d)(c+d) = 81$. If *a, b, c, d* are distinct positive integers, then $1 \cdot 81 = 81$ or $3 \cdot 27 = 81$. Let $c - d = 1$ and $c + d = 81$. Then $c = 41$ and $d = 40$. Let $a - b = 3$ and $a + b = 27$. Then $a = 15$ and $b = 12$. So $a + b + c + d = 108$.