Algebra II
State Mathematics Contest Finals
May 1, 2003

1. If m is a positive real number, determine the sum of the roots of the equation $(2x-3)^2 - m = 0$.

   a. 1 b. 3 c. 5 d. 7 e. 9

2. Simply: $\sqrt[x]{\sqrt{\sqrt[3]{25}}}$ where x is a natural number greater than 1.

   a. 25^{3x} b. $25^{\frac{1}{3x}}$ c. $5^{\frac{1}{3x}}$ d. 5^{3x} e. none of these

3. The set of real numbers satisfying $\frac{1}{x+1} > \frac{1}{x-2}$ is:

   a. $\{x \mid x > 2\}$ b. $\{x \mid -1 < x < 2\}$ c. $\{x \mid x < 2\}$
   d. $\{x \mid x < -1\}$ e. $\{x \mid x > -1\}$

4. Solve for x: $9^{x+1} + 9^{x+2} + 9^{x+3} + 9^{x+4} + 9^{x+5} = 22143$

   a. $-\frac{1}{2}$ b. $\frac{1}{2}$ c. $-\frac{1}{3}$ d. $\frac{1}{3}$ e. none of these

5. How many real solutions does the following equation have?

   $$\sqrt{1+x+\sqrt{x}} = \sqrt{x+\sqrt{x+7}}$$

   a. 4 b. 3 c. 2 d. 1 e. 0

6. A bag contains a number of marbles of which 80 are red, 24 are white, and the rest are blue. If the probability of randomly selecting a blue marble from this bag is $1/5$, how many blue marbles are there in the bag?

   a. 25 b. 26 c. 27 d. 28 e. 29

7. If $f(x) = f(x-2) + x$, and $f(7) = 11$, find $f(5)$.

   a. 10 b. 8 c. 6 d. 8 e. 4

8. When $x^4 - 6x^2 + 5$ is factored completely with integer coefficients, then the sum of the factors is:

a. $x^2 + 2x - 5$ b. $x^2 + 2x + 3$ c. $x^2 - 7$
d. $x^3 - 5$ e. none of these

9. Let $\frac{2x-11}{x^2-5x-14} = \frac{B}{x-7} + \frac{C}{x+2}$ be an identity in x. The value of B + C is:

a. 4 b. -2 c. 5 d. 2 e. -4

10. Find, if possible, the inverse of $f(x) = \frac{x+2}{x-3}$.

a. $\frac{6x+2}{x-1}, \quad x \neq 1$ b. $\frac{3x-2}{x+2}, \quad x \neq -2$ c. $\frac{3x+2}{x-1}, \quad x \neq 1$
d. $\frac{6x-2}{x+2}, \quad x \neq -2$ e. no inverse exists

11. Simplify the following rational function. $\frac{x^3 - y^3}{x^4 + x^2y^2 + y^4}$

a. $\frac{x-y}{x^2+xy+y^2}$ b. $\frac{x-y}{x^2-xy+y^2}$ c. $\frac{1}{x^2-xy+y^2}$
d. $\frac{1}{x^2+y^2}$ e. none of these

12. Determine the range of the function $F(x) = \frac{x+3}{x}$.

a. $\{x \mid x \in \text{Re als}\}$ b. $\{x \mid x \in \text{Re als}, x \neq 0\}$ c. $\{x \mid x \in \text{Re als}, x \neq -1\}$
d. $\{x \mid x \in \text{Re als}, x \neq -3\}$ e. $\{x \mid x \in \text{Re als}, x \neq 1\}$

13. If $\frac{9 + 3^{2x}}{10} = 3^x$, then the value of $x^2 + x + 1$ is

a. 0 or 2 b. 0 only c. 1 or 7 d. 7 only e. 1 only

14. Find the sum of all real solutions to the equation $y^2 = |5 - 4y|$.

a. 0 b. 1 c. 2 d. 3 e. none of these

15. For all real numbers p, q, x, y which satisfy $x > p$ and $y > q$, then which of the following inequalities are satisfied?

i. $x^2y^2 > p^2q^2$ ii. $x + y > p + q$ iii. $x^2 + y^2 > p^2 + q^2$

a. i, ii, iii b. i only c. ii, iii d. ii only e. none of these

16. Josh and nine of his friends volunteered to help clean Mr. Camp's vacant lot. Mr. Camp needed 2 mowers, 5 twig collectors and 3 to rake. In how many ways can these jobs be assigned to Josh and his friends?

a. 5040 b. 50,400 c. 15210 d. 25,200 e. 2520

17. Mickey and Minnie have x one dollar bills. Minnie noticed that if she stacked the dollar bills in stacks of 8, she had 7 bills left over. When Mickey made stacks of 6, he had 3 left over. Together they made stacks of 7 and had 4 left over. If they have less than $100, how many bills did they have left over when they made stacks of 5?

a. 4 b. 3 c. 2 d. 1 e. 0

18. The graph of two parabolas $y = 2x^2$ and $y = x^2 + x + 6$ intersect in two points. An equation for the line that passes through these two points is

a. $x - 2x + 18 = 0$ b. $2x - y - 18 = 0$ c. $2x - y + 12 = 0$
d. $2x - y + 4 = 0$ e. $x - 2y + 12 = 0$

19. Find the sum of the real roots of the equation, $6x^2 + 11x + k = 0$, if $3x - 2$ is a factor of $6x^2 + 11x + k$.

a. $-7/15$ b. $7/12$ c. $-5/3$ d. $11/3$ e. $-11/6$

20. If $f(x) = \left(\dfrac{x^4 - x^3 + x^2 - x + 1}{x}\right)^3$ and $i = \sqrt{-1}$, then $f(i)$ equals

a. i b. -1 c. $-i$ d. 1 e. 2i

21. Which of the following is equivalent to $-\log_2\left(x - \sqrt{x^2 - 1}\right)$?

a. $\dfrac{1}{2}\log_2\left(\dfrac{x^2 - 1}{x}\right)$ b. $\left(\log_2 x - \dfrac{1}{2}\log_2(x^2 - 1)\right)^{-1}$ c. $\log_2\left(x + \sqrt{x^2 - 1}\right)$
d. $\frac{1}{2}\log_2(x^2 - 1) - \log_2 x$ e. none of these

22. If $\ln \sqrt[a]{e^5} + \frac{2\ln \sqrt[b]{e^4}}{a} - \frac{5\ln \sqrt[a]{e^3}}{3} = 16$ where a and b are natural numbers greater than 1, then the value of ab is:

a. $\frac{1}{2}$ b. $\frac{1}{4}$ c. 4 d. 6 e. can not be determined

23. If $8 \cdot {}_nP_3 = 2 \cdot {}_nP_4$, then

a. $n < 6$ b. $6 < n < 10$ c. $10 < n < 20$
d. $n > 20$ e. impossible problem

24. Let F be a function for which $F(x/2) = x^2 + x + 3$. Find the sum of all values of w for which $F(3w) = 9$.

a. $-\frac{1}{3}$ b. $\frac{1}{9}$ c. $-\frac{1}{6}$ d. $-\frac{2}{3}$ e. 6

25. Suppose F(5) = 7. Find the coordinates of the corresponding point (x, y) on the graph of $y = 3 \cdot F(2x + 5) - 5$. Then $x + y$ equals

a. 15 b. 16 c. 12 d. −13 e. −16

26. An isosceles right triangle region of area 36 is cut from a corner of a rectangular region with sides of length $6\sqrt{2}$ and $6(\sqrt{2} + 1)$. What is the perimeter of the resulting trapezoid?

a. 36 b. $18\sqrt{2} + 18$ c. 30
d. $12\sqrt{2} + 24$ e. $24\sqrt{2} + 12$

27. If $\log_5 \sqrt{2} = x$, then $\log_{\sqrt{2}} 5$ equals:

a. $-x$ b. x^2 c. $-\sqrt{x}$ d. $1/x$ e. $\sqrt{x}$

28. If $x^2 - (1 - 2i)x = (\frac{1}{2} + i)$, find the complete solution.

a. $\left\{\frac{1 \pm 2i}{2}\right\}$ b. $\left\{\frac{1 \pm 3i}{3}\right\}$ c. $\left\{\frac{1+i}{i}, \frac{1+3i}{2}\right\}$
d. $\left\{\frac{1-2i}{2}, \frac{1+3i}{2}\right\}$ e. none of these

29. Simplify: $x^{\frac{1}{2}} \cdot x^{-\frac{1}{4}} \cdot x^{\frac{1}{8}} \cdot x^{-\frac{1}{16}} \cdots$

a. x b. $\sqrt{x}$ c. $\sqrt[3]{x}$ d. x^2 e. $\sqrt[5]{x}$

30. If the following system of equations

$$\begin{aligned} y &= -x + 5 \\ kx + y &= 17 \\ x + ky &= -2 \end{aligned}$$

is consistent and k is a constant, then the value of 2k + 1 is:

a. 2 b. 6 c. 4 d. 3 e. 5

31. Given $f(x) = x + \frac{1}{4}$ and $h(x) = x^{\frac{3}{2}}$, find $(h \circ f^{-1})(\frac{1}{2})$.

a. $\frac{1}{2}$ b. $-\frac{1}{4}$ c. $\frac{1}{8}$ d. $\frac{1}{4}$ e. $-\frac{1}{8}$

32. The natural numbers are grouped as indicated: {1}, {2,3}, {4, 5, 6}, {7, 8, 9, 10}, ⋯ with *m* numbers in the m^{th} group. Find the sum of the numbers in the 110^{th} group.

a. 665,555 b. 55,000 c. 55,555 d. 450,000 e. 700,565

33. Let $P(a,b)$ and $Q(c,d)$ denote two distinct points on the graph of $f(x) = x^2$. Suppose that the slope of line PQ is 5 and the x coordinates of P and Q differ by 1. Find $b + d$.

a. 41 b. 25 c. 13 d. 5 e. none of these

34. Let ***L*** denote the line which passes through the point $(7,1)$ and the center of the circle $x^2 + y^2 - 10x + 6y + 9 = 0$. Which of the following points is also on the line ***L***?

a. $(5,-7)$ b. $(4,-10)$ c. $(-3,7)$ d. $(8,3)$ e. none of these

35. What is the sum of the squares of the real and complex solutions of $x^4 + 2x^3 + 9x^2 + 18x = 0$

a. -14 b. 8 c. 5 d. -7 e. -2

36. Given: $ax + by = (a-b)^2$ and $ax - by = a^2 - b^2$.
Determine the difference in x and y.

a. $3(a+b)$ b. $3(a-b)$ c. $2(a+b)$ d. $2(a-b)$ e. $(a-b)$

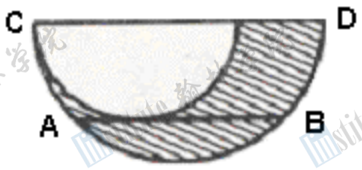
37. If the system of equations

$$y = 7\sin x + 3\cos x$$
$$y = 7\cos x + 3\sin x$$

is solved simultaneously for $0 \le x \le p$, the value of y must be:

a. $4\sqrt{2}$ b. $2\sqrt{5}$ c. 2 d. $5\sqrt{2}$ e. -2

38. Pictured are two semicircles. $\overline{AB}$ is tangent to the smaller semicircle and parallel to $\overline{CD}$. If $\overline{AB}$ is 16, find the area of the shaded region.



a. $16p$ b. $32p$ c. $8p$ d. p e. $18p$

39. Solve the following inequality. $\dfrac{(x^2 - x - 6)}{x - 5} \ge 0$

a. $(5, +\infty)$ b. $[-2, 3] \cup [5, +\infty)$ c. $[-2, 3]$
d. $(-\infty, -2] \cup [3, 5)$ e. $[-2, 3] \cup (5, +\infty)$

40. Let a, b, c, and d represent four distinct positive integers where $a^2 - b^2 = c^2 - d^2 = 9^2$.
Find the value of $a + b + c + d$.

a. 27 b. 65 c. 98 d. 108 e. 111

Algebra II

Answers

1. b		11. b		21. c		31. c
2. c		12. e		22. a		32. a
3. b		13. c		23. b		33. c
4. a		14. e		24. c		34. d
5. d		15. d		25. b		35. a
6. b		16. e		26. d		36. d
7. e		17. a		27. d		37. d
8. a		18. c		28. e		38. b
9. d		19. e		29. c		39. e
10. c		20. a		30. e		40. d

Tie Breakers

Best of three: 4, 13, 29

Sudden Death: 14, 19, 21, 28, 32, 40