ute the the 'S PL State Mathematics Contest: Geometry May 3, 2001 Solutions

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Institute # # 13 PR (b) In an equilateral triangle, the centroid, as well as the incenter and circumcenter, is located two-thirds of the distance from the vertex to the opposite side. Then $(2x)^2 = x^2 + 3^2 \Rightarrow 3x^2 = 9 \Rightarrow x = \sqrt{3}$. This makes the area $\frac{1}{2}(2\sqrt{3})3 = 3\sqrt{3}$.

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Placing the runners on a number line, we see х (b) that there is a spread of 20 seconds from first to last, stitute # # # K multilite # ** that Jo was at the end, Sam in second 19 second ahead of Jo and one second behind Chris, the winner. Pat then was in third 12 seconds behind Sam.

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3. (d) In the figure, we see that
$$CE = DE = 3\sqrt{2}$$
. Also
 $FD = BC + CE = 12 + 3\sqrt{2}$ and $AF = 5 - 3\sqrt{2}$. Thus the distance we want,
 $AD = \sqrt{AF^2 + FD^2} = \sqrt{\left(12 + 3\sqrt{2}\right)^2 + \left(5 - 3\sqrt{2}\right)^2} \approx 16.26$

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Points on both circles must satisfy both equations, so both (1) (a) matinu # '& K subtracting gives $(x^2 + 6x + 9 + y^2 - 2y + 1) - (x^2 - 2x + 1 + y^2 + 4y + 4) = 16 - 9$, or simply $8x - 6y = 2 \Rightarrow 4x - 3y = 1$

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The new length and width are 400 $(400+2w)(300+2w) = 2(400)(300) \Rightarrow$ $120000+1400w+4w^2 = 2400c$ The new length and width are 400 + 2w and 300 + 2w, so the new area is 面动机机称林塔然 Institute \$5 H 3 PE inter the the the the (050)2

$$w = \frac{-350 + \sqrt{(-350)^2 - 4(-30000)}}{2} \approx 71.22$$

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6. % % w, itillt m 林 後 % (c) A good guess, since the hypotenuse is 13, would be the 5-12-13 right triangle. The area is indeed 30 and the perimeter is 30. If you did not guess this,

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 $a^2 + 2ab + b^2 = 169 + 120 = 289$ and $a^2 - 2ab + b^2 = 169 - 120 = 49$. Thus $(a+b)^2 = 289 \Rightarrow a+b = 17$ and $(a-b)^2 = 49 \Rightarrow a-b-7$ The that a = 12 and b = 5that a = 12 and b = 5.

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clockwise, to be equal for both hands. The minute hands starts at an angle of zero and rotates at 1/60 revolutions per minute. The hour hand which have of a revolution, moves at 1/720 revolutions per minute. (It takes 12 hours times 60 minutes to make one complete revolution.) Setting these equal we have 如此称林荡院 $\frac{1}{60}t = \frac{1}{4} + \frac{1}{720}t \Longrightarrow 12t = 180 + t \Longrightarrow 11t = 180 \Longrightarrow t = 16.\overline{36}.$

(a) Let the sides have length x and 4.5. Then we have
$$4.5x = 2(2x+9)$$
, so $0.5x = 18 \Rightarrow x = 36$.

面站加坡新林塔梯 The circumference of the can us $c\mathbf{p} = 2\mathbf{p}r$, so the radius is $r = \frac{c}{2}$. This 10. % (b) side of the square piece is the same as the circumference. The volume is

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7.5

$$V = \boldsymbol{p}r^{2}h = \boldsymbol{p}\left(\frac{c}{2}\right)^{2}(\boldsymbol{p}c) = \frac{\boldsymbol{p}^{2}c^{3}}{4}$$

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withit the start (a) By the power of a point formula, we know that $x^2 = 1.2 \cdot 7.5 = 9 \implies x = 3$

(d) The area of a rectangle is base times mutilite # # 3 PS withthe the the the the height. In the figure we see that the base is Asitute to the stitute the the

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stitute # # 18 The segments joining the centers will be formed by the radii, so the sides (e) of the triangle are 3, 4, and 5 units long, making it a 3-4-5 right triangle with are 6.

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(e) The are 2 ways to get to point B, so we can start at B and then double our answer. A tree diagram shows there are 8 ways to get from B to S without Institute # # 'E ! mutale # # 3 PE matitule ## # 'S PR visiting any point twice. ₩**B** mistitute ## the the the

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19% K mutute # # 3 PE The radius is 7, so the circumference is 14p. Divide this by 5 to get 2.8p(c)The perimeter is $2a + 2b = p \Rightarrow a + b = \frac{p}{2}$. Squaring we have for the arc length. 20. (b) $(a+b)^2 = \left(\frac{p}{2}\right)^2 \Leftrightarrow a^2 + 2ab + b^2 = \frac{p^2}{4}$. But the diagonal $c^2 = a^2 + b^2$, so subtracting we get $2ab = \frac{p^2}{4} - c^2 \Rightarrow ab = \frac{p^2 - 4c^2}{8}$. 而时间他都林道像 mailule # ** 21. (d) The volume of the original solid is 200 cubic units. When two sides are 面站抽曲新林塔梯 $2 \cdot 5^2 + 4 \cdot (5 \cdot 8) = 210$. The new surface area will be $2 \cdot 10^2 + 4(10 \cdot 8) = 520$. The ratio of the surface area will be $2 \cdot 10^2 + 4(10 \cdot 8) = 520$. The doubled, the volume goes to 800. The original surface areas was ratio of the volumes is $\frac{520}{210} = \frac{52}{21} = 2\frac{10}{21}$. P P In the figure we see that triangle (b) Astitute # # 3 В MBN is similar to triangle ORN, so $\frac{BM}{BN} = \frac{OR}{RN} \Rightarrow \frac{3}{2} = \frac{7.5}{RN} \Rightarrow RN = 5.$ R М Similarly, triangle NQO is similar to

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 $\frac{NQ}{QO} = \frac{PD}{DO} \Rightarrow \frac{7.5}{5} = \frac{5.4}{DO} \Rightarrow DO = 3.6.$ Thus the length of 1 Thus the length of the rectangle is 10.6.

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(c) diagonal AC and look at triangle APC. The length of AC is $2\sqrt{2}$, so that makes triangle APC an isosceles right triangle and the altitude is $\sqrt{2}$.

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The old area was $9w^2$ and the perimeter was 20w. The new area will be (a) 而时间很新林塔梯 alitute # # 13 PE $18w^2$. The perimeter stays the same so the new length and width, L and W are such that L = 10w - W, and $(10w - W)W = 18w^2$. This is quadratic in W, so $W^2 - 10wW - 18w^2 = 0 \Rightarrow W = \frac{10w \pm \sqrt{100w^2 - 72w^2}}{2}$. This simplifies to

 $W = (5 \pm \sqrt{7}) \Rightarrow L = (5 \mp \sqrt{7}).$ Taking the length to be the longer side, the ratio is $\frac{5 + \sqrt{7}}{5 - \sqrt{7}}.$ 而如此他称林塔梯 mutitute # # 3 stilute # #

Each of the five statements involves only three statements. We will call (e) these C for the Car is locked, T for the Tickers are stolen, and W for the wallet is stolen. A truth table of all 8 possibilities is shown here:

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、饭水"	С	w.L.	W	~C	$\sim C \rightarrow W$	₩ [™] T	$W \cap \sim T$	$T \to (\sim C \to W)$	52
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happens in row 4, where Ed makes the false statement and the others make true ones. 26. (d) Opposite and the interval Since we are told that exactly one of the statements is false and the rest true, this only happens in row 4, where Ed makes the false statements

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Opposite angles in an inscribed quadrilateral must be supplementary, since 26. (d) the arcs they subtend form the entire circle. Thus the measure of the angle at vertex C 而时间他就被任务 withthe the the 's the 而出往明日新林塔院 Astitute # # 18 must be 144 degrees. un. 茶茶林 myitte # #

The surface area to paint is $2(8 \cdot 20) + 2(8 \cdot 14) - 2(3 \cdot 7) - 4(6 \cdot 5) = 382$. 27. (e) Since we need two coats, we need enough paint to cover 764 sq feet, or 1.91 gallons.

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28. (d)In the figure we see the slopes and lengths of the sides of this triangle. Since two of the slopes are negative reciprocals, and since the converse of the Pythagorean theorem applies, we do have an isosceles right triangle.

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29. (d) The four possible figures are shown here. Note that the problem does

而如此他称林塔除 not say that one of the angles of the quadrilateral has to be a right angle, only that two of the sides meet at a righ angle. The allows the one concave polygon. institute \$



Multille # # 3 PE stillte # # 13 18 (c) The sides of the new triangle will be 8, 10 and 12. Thus the perimeter is 30. 30 and the semiperimeter is 15. Using Heron's Formula, the area is

$$\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{15 \cdot 7 \cdot 5 \cdot 3} = 15\sqrt{7}$$
.

柳林戏帆 (e) Each angle of a regular n-sided polygon is $\frac{(n-2)180}{n}$, so in this case each Thillule # # 'S

angle is $\frac{(12-2)180}{12} = 150$.

Since each of the triangles are similar, ratios of corresponding sides leads us to conclude that $x^6 = 6$, so $x = 6^{\frac{1}{6}}$. The area 32. the triangles are $\frac{1}{2} \cdot 1 \cdot x \cdot \sin(60^\circ)$ and

 $\frac{1}{2}x^5 \cdot x^6 \cdot \sin(60^\circ)$, so the ratio is of their areas is $=6^{5/3}$

$$\frac{\frac{1}{2}x^{11}\sin(60)}{\frac{1}{2}x\sin(60)} = x^{10} = \left(6^{\frac{1}{6}}\right)$$

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33. (e) The cube can be flattened to look like the figure, showing that G is on the opposite side of the 加加加加林塔然 withthe \$6 # 'S PE withit the start 's block from A. stitute ##

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34. ¹⁶ (c) 柳林浩豫 34. (c) The radius of the largest circle is $\frac{a+b}{2}$ while the radii of the two smaller circles are $\frac{a}{2}$ and $\frac{b}{2}$. Thus the area of the shaded region is $\frac{p}{2}\left(\left(\frac{a+b}{2}\right)^2 - \left(\frac{a}{2}\right)^2 - \left(\frac{b}{2}\right)^2\right)$. This simplifies to $\frac{\mathbf{p}}{2}\left(\frac{a^2+2ab+b^2-a^2-b^2}{4}\right) = \frac{\mathbf{p}ab}{4}$. 35 (a) If the diameter of d If the diameter of the wheel is 70cm, the circumference is 70pcm. to $120km \cdot 1000 \frac{m}{km} \cdot 100 \frac{cm}{m} = 12000000cm$ yields about 54567. Dividing this into $120km \cdot 1000 \frac{m}{km} \cdot 100 \frac{cm}{m} = 120000000cm$ yields about 54567.4

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surface area is proportional to the side squared, so the surface area is proportional to the volume raised to the two-thirds power $SA = hV^{\frac{3}{2}}$ and 36. surface area will be $(2V)^{\frac{2}{3}} = 2^{\frac{2}{3}}V^{\frac{2}{3}}$.

37. As shown in the figure, the midpoint of the diameter becomes the vertex (a) of the cone and points A and B meet on the circumference of the base of the cone. The radius OA becomes the slant height

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circumference of the semicircle becomes the circumference of the

cone. So $6\mathbf{p} = 2\mathbf{p} r$, where r is the radius of the cone. The radius is then 3, the slant height 6, so the

altitude is $\sqrt{6^2 - 3^2} = \sqrt{27} = 3\sqrt{3}$

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Thus the volume of the cone is $\frac{\mathbf{p}}{3}r^2h = \frac{\mathbf{p}}{3} \cdot 3^2 \cdot 3\sqrt{3} = 9\mathbf{p}\sqrt{3}$. tt en the second states of the second re 新林塔

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with the star to the start mistitute ### (d):111 Through the center of the circle O, drop the 38. perpendicular to the tangent line. This will have length 18, and will include the radius of length 13 and segment OB of length 5. By the Pythagorean theorem, AB = 12 and the chord has length 24, since the segment perpendicular to the chord also bisects it. 而此此他新林塔院 而如此他都林道熊 Institute # # '& PR Withte the the 's PR Astitute ## # '& PR 面站加坡新林塔梯

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