Johns Hopkins Math Tournament 2021

Proof Round A - Middle School

Game Theory

April 3rd, 2021

Instructions

- To receive full credit, answers must be legible, orderly, clear, and concise.
- Even if not proven, earlier numbered items may be used in solutions to later numbered items, but not vice versa.
- While this round is asynchronous, you are still NOT ALLOWED to use any outside resources, including the internet, textbooks, or other people outside your teammates.
- Put the **team number** (NOT the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.
- To submit your answers, please email ONE SINGLE PDF containing all your answers to jhmt2021proofroundA@gmail.com, with the subject tag as "Team # Proof Round A". For example, if your team number is 0, then the subject should be "Team 0 Proof Round A".

1 What is Game Theory?

Intuitively, all of us know what a game is, and know of examples of games. From video games to tag to Tic-Tac-Toe, all games share common traits, such as a set of rules that dictate how the game is played and different outcomes for players that help to determine who "wins" and who "loses."

Definition 1.1. For the rest of this round, we will define a **game** as formally having the following components:

- **Players** who participate in the game.
- **Rules** that dictate what is allowed and when the game ends.
- **Actions/Moves** that the players undertake.
- **Outcomes** for each player when the game ends. Each player aims to achieve the best possible outcome for themselves.

Remark 1.1. *We will also assume that players never cheat in games (just like you should never cheat on this round!), and unless otherwise specified, players always play optimally.*

Example 1.1. *In Tic-Tac-Toe, there are two players whose actions are placing either X's or O's on a* 3×3 *grid. The rules are that the players alternate turns, and whoever connects* 3 *of their symbol in a row either horizontally, vertically, or diagonally first wins. The outcome is either one playing winner and one losing, or the game resulting in a tie if the board is filled without any* 3 *symbols in a row.*

Problem 1: In each of the following activities, clearly state what the players, actions/moves, and outcomes are. Note for some of these, there may not be one specific answer, so feel free to include justifications.

(a) (2 points) Chess
(b) (2 points) Connect 4 (US Version) / Mahjong (ASDAN Version)
(c) (3 points) An auction
(d) (3 points) Symbiotic relationships (ex. clownfish and sea anemone)

(a) We were fairly lenient on grading. As long as the answer contained some mention of the general rules of Chess (specific details were not required), the fact that there are two players, and that games end in a win/draw/lose for each player, we gave points.

(b) Again, we were fairly lenient on grading. As long as the answer contained some mention of the general rules of Mahjong, the fact that up to 4 players can play, and that games end in a winner, we gave points.

(c) There were many acceptable answers to this problem, we were just checking that students gave a somewhat reasonable argument as to how an auction can be a game.

One example of an acceptable answer was to say that the bidders were the players, the rules were standard auction rules, and the outcomes were whether or not each player won the auction.

(d) Again, there were many acceptable answers to this problem.

One example of an acceptable answer was to say that the participants in the relationship (ex. the clownfish and the sea anemone) were the players, the rules were related to what each player did for the other(s), and the outcomes were survival/death.

2 Common Knowledge

Definition 2.1. For a group of actors, **common knowledge** of some information exists when all actors know that information, all actors know that the other actors know that information, all actors know that the other actors know that they themselves know that information, etc.

Example 2.1. *If two players are competing in a game of Chess, common knowledge encompasses everything that is currently on the board, including the current position of all pieces in play, what pieces have been captured and are not in play, and which player's turn is next. Common knowledge* ***does not*** *emcompass each player's strategy, as neither player is guaranteed to know exactly what the other player plans on doing next.*

Common knowledge is a key element of playing any game, as players often must consider what other players know when evaluating what their best move is. The problem below is an example of a situation in which the common knowledge is increasing as each player reveals what they know.

Problem 2: Kevin chooses two numbers a and b from the integers between 1 and 6 inclusive, such that $a \leq b$. He then tells Bill only the product ab, and Jack only the sum $a+b$. Note: for both parts, assume that both Jack and Bill are able to make perfect inferences from each other's statements. You do not need to justify your answers, just list the (a, b) pairs.

(a) (2 points) If Bill says "I know what a and b are," then how many possible values of (a, b) exist?

(b) (8 points) Finally, for what pairs (a, b) would Bill and Jack have this conversation:

- Bill: I don't know what a and b are.
- Jack: I also don't know what a and b are.
- Bill: I now know what a and b are.

(a) $\boxed{15}$. If Bill immediately knows what a and b are, then it must be the case that given the product ab, there is only one pair (a, b) that can produce the given product. Below is a list of the 15 possible values of (a, b):

- $(1,1)$
- $(1,2)$
- $(1,3)$
- $(1,5)$
- $(2,4)$
- $(2,5)$
- $(3,3)$
- $(3,5)$
- $(3,6)$
- $(4,4)$
- $(4,5)$
- $(4,6)$
- $(5,5)$
- $(5,6)$
- $(6,6)$

(b) The only two answers are $(1,4)$, $(3,4)$. First, we exclude all 15 possible (a,b) pairs with $a \leq b$ mentioned in part a above. Thus, we are left with 6 possible pairs, listed below.

- $(1,4)$
- $(1,6)$
- $(2,2)$
- $(2,3)$
- $(2,6)$
- $(3,4)$

Now, consider the pair $(2,6)$ from Jack's perspective. Jack has the number 8, so from the beginning, he knows that the possibilities for (a,b) are $(2,6)$, $(3,5)$, and $(4,4)$. However, once Jack hears that Bill does not know what a and b are, Jack can deduce that $(3,5)$ and $(4,4)$ are no longer possibilities, and thus the answer must be $(2,6)$. However, since in the actual conversation Jack still does not know what a and b are after Bill's statement, $(2,6)$ is not a possible pair. A similar argument can be made for $(2,2)$, which means we have 4 possibilities left.

- $(1,4)$
- $(1,6)$
- $(2,3)$
- $(3,4)$

Now suppose the pair was $(1,6)$. Bill would be told the number 6, so from his perspective the possible values of a and b are $(1,6)$, and $(2,3)$, so he does not know what a and b are immediately. From Jack's perspective, the possible values of a and b are $(1,6)$, $(2,5)$, and $(3,4)$. Since Bill does not immediately know what a and b are, Jack can eliminate $(2,5)$ as a possibility, but Jack still doesn't know how to decide between $(1,6)$ and $(3,4)$, so he also still does not know what a and b are.

Now Bill again considers the two possible options from his perspective: $(1,6)$ and $(2,3)$. Unfortunately, Bill realizes that in both cases, Jack would not be able to figure out a and b from his initial statement (i.e. Bill's initial statement of "I don't know"), and so even after Jack's reply, Bill still can't distinguish between these two possibilities. As a result, $(1,6)$ is not a possible answer.

Finally, suppose the pair was $(1,4)$. Bill would be told the number 4, so from his perspective the possible values of a and b are $(1,4)$ and $(2,2)$, so he does not know what a and b are immediately. From Jack's perspective, the possible values of a and b are $(1,4)$, and $(2,3)$. Jack can't eliminate either of these options from being Bill's initial statement, so he also still does not know what a and b are.

Now, Bill does casework on $(1,4)$ and $(2,2)$. However, the difference here is that if the answer was $(2,2)$, Jack WOULD FIGURE OUT that a and b are $(2,2)$, because Jack would be able to eliminate $(1,3)$ from Bill's first response. Thus, Bill can successfully conclude that the answer is $(1,4)$.

Finally, if we repeat the logic above on $(2,3)$ and $(3,4)$, we find that $(3,4)$ is a possible answer while $(2,3)$ is not.

Problem 3: Triplets Albert, Brian, Clyde play a game where one of the three always tells the truth, another always tells a lie, and the third randomly says either a truth or a lie. All three know each other's roles, and they now challenge you to also deduce the roles of each. In the following situations, state the role of each person:

(a) (2 points) Albert says "I always lie". Clyde says "Albert always tells the truth".

(b) (3 points) *Note that this problem was not on the ASDAN test.* Brian says "Clyde never tells the truth". Albert responds "Brian's statement is a lie".

(c) (4 points) *Note that this was problem (b) on the ASDAN test.* Albert says "Brian is the one who always says something random". Clyde then says "Albert always lies". Finally, Brian then says "enough has been said to determine each role".

(a) Albert is random, Brian always tells the truth, Clyde always lies. First, consider Albert's statement of "I always lie". Given this statement, Albert cannot always tell the truth (as then this statement would contradict the truth), and Albert cannot always tell a lie (as then this statement would not be a lie). Thus, Albert must be the random. Given this, we find that Clyde's statement is a lie, and thus Clyde must be the liar. Lastly, Brian must tell the truth because it is the only role remaining.

(b) There were multiple possible answers for this problem. We accepted RTL, TLR, RLT, TRL as roles for Albert, Brian, and Clyde respectively, where R represents random, T represents truth teller, and L represents liar. Refer to solution technique described in part (c) below for examples on how to get to these solutions.

(c) Albert always tells the truth, Brian is random, Clyde always lies. This problem is a bit trickier than part a, so let us consider all $3 \times 2 \times 1 = 6$ possible ways the roles can be distributed among Albert, Brian, and Clyde. Given Albert's statement, it is not possible for Albert to be the truth teller and Brian to be the liar, and it is not possible for Albert to be the liar and Brian to be random. Thus, after Albert's statement, there are $6 - 2 = 4$ possibilities left for roles:

- Albert always tells the truth, Brian is random, and Clyde always lies.
- Albert always lies, Brian always tells the truth, and Clyde is random.
- Albert is random, Brian always tells the truth, and Clyde always lies.

Next, we analyze Clyde's statement that "Albert always lies". Among the four remaining possibilities above, we can remove the case where Albert is random and Clyde always tells the truth, because Clyde's statement would be a lie. Thus we now have 3 remaining possibilities for roles. Finally, if we look at Brian's statement that "enough has been said to determine each role", we find that this statement cannot be the truth because we in fact *do not* have enough information yet to determine each role. Thus, we can remove the two remaining role assignments in which Brian tells the truth. At this point, there is only one possibility left, which is the one where Albert always tells the truth, Brian is random, and Clyde always lies.

Lastly, note that on the US test, we also accepted RLT as an answer due to unclear wording in the original problem.

3 Nash Equilibrium

Golden Balls is a game show where two contestants compete for some amount of money, say $\$100,000$. Both contestants must choose to either "steal" or "split." If both choose to split, then both will walk away with half the money, or $\$50,000$ each. However, if one contestant chooses "steal" while the other chooses to "split," the contestant that chose "steal" will take the entire $\$100,000$. Lastly, if both contestants choose to steal, then neither will get anything.

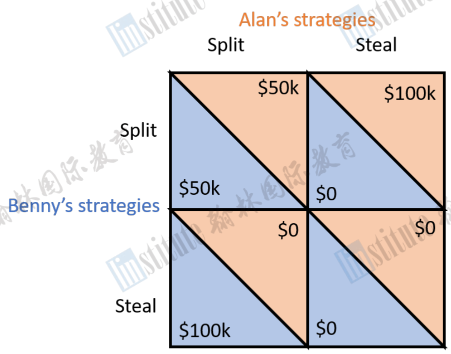
If we were a contestant on *Golden Balls*, what would our best move be? Clearly, we want to consider what actions our opponent can take, but due to common knowledge, we also know that our opponent is considering the move we take. As a result, is one option definitively better than the other?

Definition 3.1. In a game, the **Nash Equilibrium** is the situation in which all players make a best possible action considering the actions of other players, and thus no player has any incentive to change their action.

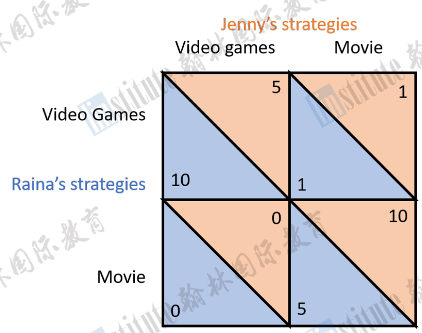
Example 3.1. *In Golden Balls, it may be surprising to see that the Nash Equilibrium occurs when both players choose to "steal," even though this clearly results in a worse outcome than if both players chose to "split." This is because once both players choose to "steal," neither has anything to gain from switching, since if either instead chooses "split" but their opponent chooses "steal," they still gain nothing. On the other hand, both players choosing to "split"* ***is NOT*** *a Nash Equilibrium because either player can improve their outcome by instead choosing to "steal." Another way to interpret this equilibrium is to only consider one participant's perspective, say Benny. From Benny's point of view, if Alan decides to "split," then it is better if he (Benny) chooses to "steal." If Alan decides to "steal," it doesn't matter what he (Benny) picks. Regardless of Alan's decision, "steal" is a best choice for Benny, but "split" is not. Alan goes through this same logic, both players will settle on "steal."*

Definition 3.2. A **payoff matrix** is a table that shows the possible actions that all players can make, as well as the payoffs associated with each one. These matrices are often used to visualize where Nash Equilibria lie.

Example 3.2. *Below is an example of a payoff matrix for two contestants, Alan and Benny, who are competing in Golden Balls.*



Problem 4: (3 points) Jenny and Raina are trying to decide on what to do this weekend. Jenny prefers watching movies while Raina prefers playing video games. However, since Jenny and Raina are friends, they also prefer to do something together rather than doing do different activities separately. Below is the payoff matrix for Jenny and Raina in this situation, where the numbers represent the happiness levels of Jenny and Raina in each situation. Are there any Nash Equilibria in this problem? If so, describe them and justify why they are indeed equilibria. If not, explain why no equilibria exist.

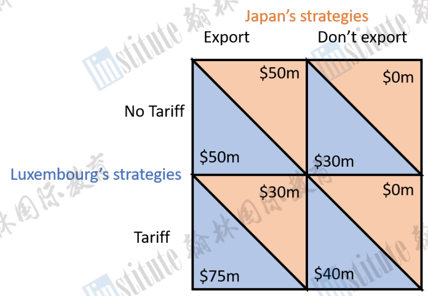


Two Nash Equilibria: both girls playing video games, and both girls watching movies.

First consider the equilibria when both Jenny & Raina play video games. From Jenny's perspective, there is no reason to switch to watching movies (even though Jenny prefers movies) because she has less enjoyment watching movies alone than she does playing video games with Raina. Similarly for Raina, her payoffs are reduced if she switches to watching movies, so Raina will also not change her decision. Since neither player will change their decision, this is a Nash equilibrium. A similar argument can also be made to show that both Jenny & Raina watching movies is also a Nash equilibrium.

Problem 5:

Japan and Luxembourg are allies who frequently trade with each other. Luxembourg realizes that Japanese anime imports are hurting local animation studios, and thus plans on imposing tariffs on Japanese imports. In retaliation, Japan is considering altogether stopping anime exports to Luxembourg. Below is the payoff matrix for this situation.



(a) (2 points) What is Japan's best action/strategy and why?

(b) (2 points) What is Luxembourg's best action/strategy and why?

(c) (1 points) What is the outcome of the game if both players play optimally?

(a) Japan's best action is to export. This is because regardless of whether Luxembourg tariffs, Japan will earn more money when they export.

(b) Luxembourg's best action is to impose tariffs. This is because regardless of whether Japan exports, Luxembourg will earn more money when they impose tariffs.

(c) Japan will export and Luxembourg will impose tariffs. This is a direct result of the findings of part a and b above.

Problem 6: Consider the game Rock, Paper, Scissors where both players choose either "Rock," "Paper," or "Scissors." "Rock" beats "Scissors," "Scissors" beats "Paper," and "Paper" beats "Rock." Suppose winning the game gives a player an outcome of 1, losing gives an outcome of -1, and tying gives an outcome of 0.

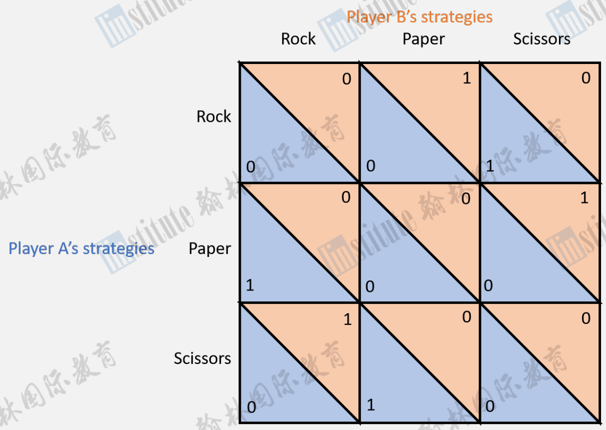
(a) (2 points) Draw the payoff matrix for Rock, Paper, Scissors between player A and player B.

(b) (2 points) Is there a Nash Equilibrium in Rock, Paper, Scissors? Explain why or why not.

(c) (3 points) Now, suppose player A and B are tired of playing classic Rock, Paper, Scissors, so they decide to add a fourth option, "Sword." "Sword" beats "Scissors" and "Paper" but loses to "Rock." Draw the payoff matrix for this modified game, Rock, Paper, Scissors, and Sword.

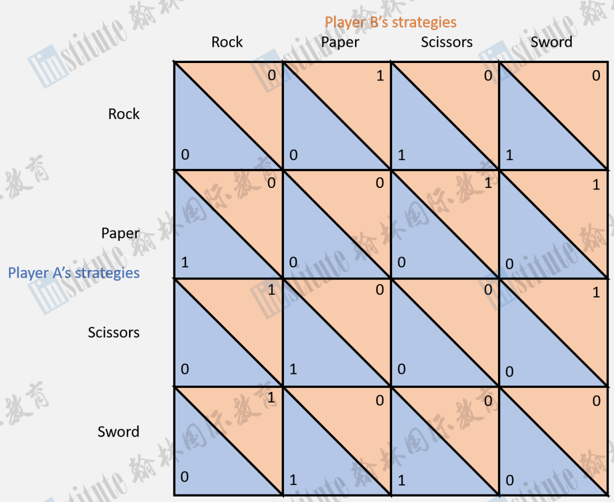
(d) (4 points) Is the game Rock, Paper, Scissors, and Sword different from regular Rock, Paper, Scissors? If so, why, and if not, why not? Remember that it is assumed that all players make optimal decisions.

(a) Refer to the image below.



(b) No, there is no Nash Equilibrium. This is because for each of the 9 squares in the payoff matrix, one of the two players (or both for the squares with ties) can improve their outcome by changing their strategy. Thus, there is no Nash Equilibrium.

(c) Refer to the image below.



(d) No, Rock, Paper, Scissors, Sword is the SAME GAME as Rock, Paper, Scissors. This is because if both players play optimally, then no player will ever choose to play "Scissors", because "Sword" is a strictly better option. However, since

"Sword" still ties against itself and loses to rock, if we imagine removing the "Scissors" row and column from the payoff matrix above, we are left with exactly the same matrix as shown in part a in the original game.

Problem 7: Tom and Jerry both own businesses selling offbrand Gucci sunglasses on the internet. Since rare items sell for more, both Tom and Jerry know that their sunglasses will sell for $\$(12-x)$ each, where x is the total number of sunglasses that are being sold by Tom and Jerry combined. For example, if Tom and Jerry both sell 1 pair of sunglasses, each will sell for $\$(12-2) = \$10$. In order to avoid complicated game theory decision making, Tom and Jerry agree with each other to sell 3 pairs of sunglasses each (i.e. they will sell a total of 6 pairs combined).

(a) (2 points) If both Tom and Jerry keep their agreement, how much money will each make and why?

(b) (3 points) If Tom decides to secretly break the agreement and start selling 4 pairs of sunglasses, how will Tom's income change and why? What about Jerry's income? Assume that Jerry sticks to the original agreement.

(c) (3 points) Does Tom have an incentive to break the agreement and start selling more sunglasses? Why or why not?

(d) (3 points) What is the Nash Equilibrium of this situation? Give your answer in terms of the number of sunglasses Tom and Jerry will sell.

(a) **Tom and Jerry will both earn \$18, or they will earn a total of \$36**. If Tom and Jerry both keep their agreement, then they will each sell 3 pairs of sunglasses, and each pair will sell for $\$(12-6) = \$6$. Thus, they will both earn $3 \times \$6 = \$18$.

(b) **Tom will earn \$20 and Jerry will earn \$15**. If Tom sells one more pair of sunglasses, then now each pair will sell for $\$(12-7) = \$5$. Thus, Tom will earn $4 \times \$5 = \$20$ and Jerry will earn $3 \times \$5 = \$15$.

(c) **Yes, because Tom earns more money by breaking the agreement**. As shown in part c above, Tom increases his earnings by \$2 when he breaks the agreement, so he is incentivized to break it.

(d) Three possible answers, full credit was given was any of them:

- **Tom sells 4 sunglasses and Jerry sells 4 sunglasses**.
- **Tom sells 5 sunglasses and Jerry sells 3 sunglasses**.
- **Tom sells 3 sunglasses and Jerry sells 5 sunglasses**.

When both Tom and Jerry are selling 4 sunglasses, each pair will sell for $\$(12 - 8) = \$4$ and thus Tom and Jerry will both earn $4 \times \$4 = \$16$. Now, if either person sells one more pair of sunglasses, each pair will sell for $\$(12 - 9) = \$3$, and they will make $5 \times \$3 = \$15$. Similarly, if either person sells one less pair of sunglasses, each pair will sell for $\$(12 - 7) = \$5$, and they will make $3 \times \$5 = \$15$. Thus, since neither Tom nor Jerry make more money by changing the number of sunglasses they sell, this is a Nash Equilibrium. A similar argument can be made to see why the other two possible answers are also equilibria.

4 Optimal Strategy

Now that we've familiarized ourselves with the concept of common knowledge and practiced some structured game theory questions involving Nash Equilibrium, let's tackle a few more general game theory problems!

Problem 8: (4 points) In the game *SPLIT*, two players take turns choosing an existing stack of Legos and splitting it into two unequal stacks. The last player that is able to make a move wins.

For example, if the game starts with a single stack of 4 Legos, then the first player's only valid move is to split the single stack into a stack of 3 Legos and another stack of 1 Lego, and then the second player's only valid move is to split the stack of 3 Legos into a stack of 1 and another stack of 2. Then, the first player cannot make a valid move, because the only remaining stacks have either 1 or 2 Legos, none of which can be split into two unequal stacks. Thus, player two will always win.

If we instead start the game of *SPLIT* with a single stack of 7 Legos, can either player guarantee that they will always win? If either player (or both) can, describe their strategy and why it guarantees a win.

The second player can always win with correct play. In order to show this, we will split into cases based on what the first player does on their first turn. Also recall that as stated in the problem, if the game ever reaches a point where the only stack that can be split is a single stack of 4 Legos, then the play who must first split the 4 Lego stack always loses.

- One possible move that first player can make is to split the 7 Lego stack into a stack 6 and a stack of 1. From this point, if the second player then splits the stack of 6 into a stack of 4 and a stack of 2, then the second player guarantees that they win. This is because the first player can now only split the stack of 4, and as mentioned above, this means they will lose.
- Another possible move that the first player can make is to split the 7 Lego stack into a stack of 5 and a stack of 2. From this point, if the second player splits the stack of 5 into a stack of 4 and a stack of 1, then they again guarantee a win because the only stack left that can be split is the stack of 4.
- The final possible move that the first player can make is to split the 7 Lego stack into a stack of 4 and a stack of 3. From this point, if the secon player splits the stack of 3 into a stack of 2 and a stack of 1, then they again guarantee a win because the only stack left that can be split is the stack of 4.

Problem 9: In the game of 71! (*Note that* 61! *was used in the US version, but all the solutions below are still applicable*), two players take turns saying an integer between 1 and 4 inclusive, and adding whatever integer they say to a shared counter (that starts at 0). The player who first has to say a number that causes the counter to reach 71 or higher loses.

(a) (1 points) Suppose Sid is playing Tyler in 71!, and Sid is going first. Sid doesn't know the strategy to the game, so he always says 4. If Tyler also decides to always say one specific number, what number(s) could he say to guarantee he wins?

(b) (3 points) Sid realizes that if he always says 4, Tyler can win every game. Thus, he decides to change his strategy to instead randomly pick 1, 2, 3, or 4 each turn. If Sid is still going first, what strategy could Tyler employ to guarantee he always wins? Your answer should specify what number Tyler should respond with regardless of which number Sid randomly picks each turn.

(c) (3 points) With your help, Tyler is still winning every game against Sid. Thus, Sid decides to change up the game so that instead of the game ending when the counter reaches 71, the game will instead end when the counter reaches a random positive integer limit that Sid chooses before each game. Will the strategy you specified in part (b) above still work regardless of which counter limit Sid chooses? If yes, explain why. If no, describe which specific counter limits would allow Tyler to still guarantee a win.

(a) Tyler can say 1 or 3. If Tyler says 1, then every time both Sid and Tyler take a turn, $4 + 1 = 5$ will be added to the counter. Thus, the counter will eventually reach 70 after Tyler's turn, and after this Sid will lose the game. A similar argument can be made to see why 3 also guarantees a win for Tyler but 2 and 4 do not.

(b) If Sid chooses x, then Tyler should always respond with $5 - x$. If Tyler employs this strategy, then every time both Sid and Tyler take a turn, $x + 5 - x = 5$ will always be added to the counter, and as a result the counter will eventually reach 70 after Tyler's turn, and after this Sid will lose the game.

Also, note that the reason Tyler must choose $5 - x$ instead of say, $7 - x$ (which at first glance may seem like an option since 7 is also a factor of 70) is because it is not always possible for Tyler to respond with $7 - x$. For example, if Sid picks 1, then Tyler cannot pick $7 - 1 = 6$ because 6 is not a possible option.

(c) Tyler can only guarantee a win if the counter limit c is such that $c \equiv 1 \mod 5$. This result is a direct result of what was already discussed in the answer to part b above. Since the possible moves for each player are only 1, 2, 3, and 4, Tyler can only guarantee that the sum of his move and Sid's move each turn is 5. Thus, Tyler can only guarantee a win if the counter is one more than a multiple of 5, because only in those situations Tyler's strategy of saying $5 - x$ always cause the counter to reach the multiple of 5 one less than c.

5 Artificial Intelligence in Games

Today, the best players in many complex and historically significant games are not humans, but computers. But how do computer algorithms know how to "play" a game? Unlike humans, algorithms can't easily be told to follow complex strategies.

Instead, algorithms oftentimes "play" games by considering every possible action it and other players can make for several turns or until the end of the game in order to determine which sequence of actions results in the optimal, or best possible outcome.

Definition 5.1. A game is said to be **solved** if the outcome of the game can always be determined regardless of the current state of the game, assuming both players play perfectly.

Example 5.1. *Simpler games such as Tic-Tac-Toe are solved. In the case of Tic-Tac-Toe, we know that the total number of possible games is at most* $9!$*, since on the first turn the first player has* 9 *possible places to place their piece, then the next player has* 8 *possible places left to place their piece, and so on. Since* $9! \approx 300,000$*, a computer is easily able to store every possible game.*

Problem 10: (3 points) In the game of *Battle!*, a standard 52-card deck plus the 2 Jokers is randomly shuffled and then evenly dealt between two players. Both players then choose one card in their hand to play (playing a card removes it from a player's hand), and whoever plays the larger card wins a point (suppose 2 is the smallest card, Joker is the largest card, and *Ace* is the second largest card). The players repeat this process until both have exhausted their deck, and then the player with the most points wins. How many ways can this game play out? Leave your answer as an unsimplified expression (ex. 9!).

$$\boxed{\binom{54}{27} \times 27! \times 27!} \text{ OR } \boxed{54!}$$

There are a total of 54 cards in the deck. First, notice that there are $\binom{54}{27}$ ways of splitting the cards into two equal halves to distribute to the two players at the beginning of the game. Then, each play can play their 27 cards in 27! ways, for a total of $\binom{54}{27} \times 27! \times 27!$ ways the game can be played out. Notice that this expression is equivalent to 54!.

Problem 11: (3 points) In the game of *Guess the Number*, Freddy secretly chooses a number between 1 and 20 inclusive without revealing it, and then asks Jethro and Insoo to guess his number. Next, Jethro guesses a number between 1 and 20, and finally Insoo also guesses a number between 1 and 20 with the condition that the positive difference between his guess and Jethro's guess must be at least 5. Whoever's guess is closest to Freddy's number wins. How many ways can this game play out? Give an exact number as your answer.

$$\boxed{4800}$$

Freddy can choose any number from 1 to 20 inclusive with no restrictions. However, we must be slightly more careful when counting the number of ways Jethro and Insoo can choose numbers due to the restriction that their guess must be at least 5 apart. Thus, we find that:

- If Jethro picks 1 or 20, then Insoo has 15 possible choices to pick from.
- If Jethro picks 2 or 19, then Insoo has 14 possible choices to pick from.
- If Jethro picks 3 or 18, then Insoo has 13 possible choices to pick from.
- If Jethro picks 4 or 17, then Insoo has 12 possible choices to pick from.
- If Jethro picks any number from 5 to 16 inclusive, then Insoo has 11 choices to pick from.

Thus, we find that the number of ways Jethro & Insoo can pick their numbers is:

$$2 \times 15 + 2 \times 14 + 2 \times 13 + 2 \times 12 + 12 \times 11 = 240$$

Thus, the answer is $20 \times 240 = 4800$.

Unfortunately, not every game can be solved by listing out every possible way the game can be played. For example, current estimates of the complexity of Chess state that there are at least 10^{120} possible ways a full game of Chess can be played, an unfathomably large number (for reference, there are only 10^{80} atoms in the universe). Despite this, starting in the 2000s, computer algorithms were easily able to defeat the best human players, and today top Chess players regularly use computer analysis to improve.

How are computers able to play Chess at such a high level despite the complexity of the game? The answer lies in a concept called pruning. Unfortunately, we won't have time to discuss pruning in-depth in this test, but the central idea behind pruning is that computers do not actually check *every single* possible way a game can play out. Instead they "prune" or remove cases where it is highly unlikely that the optimal solution will be found.

Example 5.2. *When Chess algorithms try to find the best move, they may oftentimes prune cases where a Queen (commonly thought of as the strongest piece in Chess) is sacrificed for no gain, as it is almost never the case that the optimal sequence of moves involves such a sacrifice.*

Problem 12: (2 points) Do you think that artificial intelligence algorithms that use pruning will always be able to find the optimal strategy? Why or why not?

No, because they might accidentally prune the optimal strategy.

We only accepted "no" as an answer. While pruning algorithms are usually successfully in greatly improving the efficiency of algorithms while still playing well, there is always a risk that the true optimal solution will be accidentally pruned, especially in complicated games like Chess where it is unclear whether a move that looks bad in the short-run can end up being important many moves down the line.

Problem 13: (2 points) In example 5.1 above, it was stated that there are $\approx 9!$ ways the game Tic-Tac-Toe can be played out. Suppose a computer algorithm was trying to determine the optimal solution to Tic-Tac-Toe but pruned out every game in which the first player places their first piece in the top row of the board. What is a new approximation for the number of ways Tic-Tac-Toe can be played out? Leave your answer as an unsimplified expression (ex. 9!).

$$\boxed{6 \times 8!}$$

is the most reasonable solution. This is because if we cannot place a piece on the top row for the first turn, then there are only 6 options for the first move. However, after the first move, the game is played as normal, so after the first move there are approximately 8! ways to play the rest of the game.