Johns Hopkins Math Tournament 2021
Proof Round C - High School

Chip-Firing Games

April 3rd, 2021

Instructions

- To receive full credit, answers must be legible, orderly, clear, and concise.
- Even if not proven, earlier numbered items may be used in solutions to later numbered items, but not vice versa.
- While this round is asynchronous, you are still NOT ALLOWED to use any outside resources, including the internet, textbooks, or other people outside your teammates.
- Put the **team number** (NOT the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.
- To submit your answers, please email ONE SINGLE PDF containing all your answers to jhmt2021proofroundC@gmail.com, with the subject tag as "Team # Proof Round C". For example, if your team number is 0, then the subject should be "Team 0 Proof Round C".

Introduction

In this proof round, we will learn to play some deceptively simple games on graphs known as *chip-firing games*. By the end of the round, we will be able to distinguish between games that terminate in a finite number of turns and games that can continue ad nauseum.

1 Graphs & Chips

Before we can discuss chip firing games, we need to define graphs.

Definition 1.1. A *graph* is the mathematical term for a network. It consists of a bunch of *vertices*, represented by circles, some of which are connected by *edges*, represented by lines drawn between the circles. See examples of graphs in the figures throughout this test.
Note: For this test, we will assume that there is a path of edges leading from any vertex to any other vertex in any graph.

Definition 1.2. Suppose we locate a pile of chips at each vertex in a graph. Define a *chip configuration* C on a graph G to be a list of the number of chips at each vertex in the graph. (In order to write this down, we must assign names or numbers to the vertices.) If v is a vertex in G, then we write $C(v)$ as the number of chips at v.

2 Chip Firing Games

Given a chip configuration C on a graph G, we can play a one-player game, called Chip Firing, that involves moving chips from vertex to vertex along edges in G.

Definition 2.1. A *chip firing* from a vertex $v \in G$ is an action that can be taken on the chip configuration. We change the number of chips at each vertex in G in the following way:
- $C(v) \to C(v) - d(v)$, where $d(v)$ is the number of vertices connected to v by an edge.
- For each vertex w connected to v by an edge, $C(w) \to C(w) + 1$.
- For all other vertices, the configuration is unchanged.

Essentially, the chosen vertex 'sends' one chip to each of its neighbors.

Chip firings can only be performed at vertices v where $C(v) \geq d(v)$.

Problem 1: (5 points) See the graph G below. Suppose an initial configuration C is $(0, 4, 2, 1, 1)$. This means there are 0 chips at vertex 1, 4 chips at vertex 2, etc. Perform the following sequence of chip fires on C and give the resulting final chip configuration: $2 \to 3 \to 5 \to 4 \to 2$.

Sequence is $(1, 1, 3, 2, 1)$, $(2, 2, 0, 3, 1)$, $(2, 2, 0, 4, 0)$, $(2, 3, 1, 1, 1)$, and final configuration $(3, 0, 2, 2, 1)$.

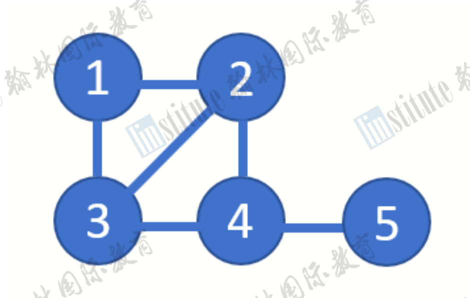

Figure 1: Question 1 graph.

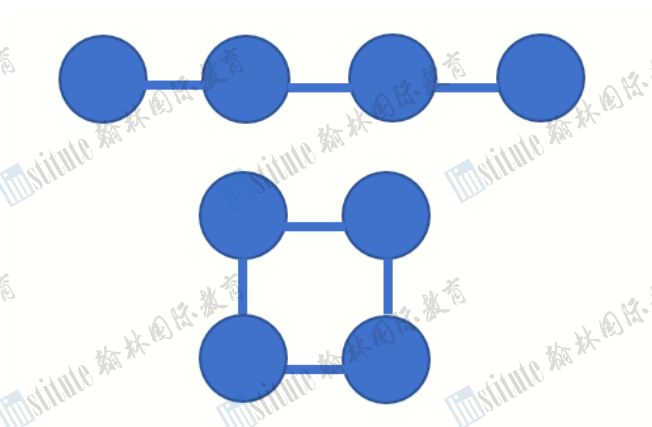

Figure 2: A path and a cycle.

Definition 2.2. A graph is called a *path* if its edges form a single line, ie. the vertices in the graph can be ordered such that the vertices are only connected to vertices next to them in the order.

Problem 2: (10 points) Suppose G is a path with n vertices and an initial configuration C on G is n chips at one of the end vertices. Suppose we fire chips until no fires are possible, with the stipulation that we never fire nodes at the ends of the path if they only have 1 chip. What is the final configuration on the graph? Prove your result for any valid sequence of fires.

Final configuration is one chip at each node. Proof by induction. Base case: 2 nodes, 2 chips, 1 fire, trival. Induction step: Suppose we have a chain of $n+1$ nodes, and assume the result for n nodes. By assumption, there is some number of fires after which the result is one chip at each node except the last, plus one extra chip still at the first node. From here, the only valid fire is the first node. The first node sends one chip to the second node, which is then the only node that can fire. The second node fires, passing the 'extra' chip forward again. This continues until the extra chip reaches the other end node, at which time we stop firing.

Definition 2.3. A graph is called a *cycle* if its edges form a single loop, ie. the vertices in the graph can be ordered such that the vertices are only connected to vertices next to them in the order, and the first vertex is additionally connected to the last vertex.

Problem 3: (5 points) Suppose G is a cycle and an initial configuration C on G is 2 chips at each vertex. Prove that, if every vertex is fired exactly once in a valid sequence, the graph returns to its initial configuration.

By assumption, such a sequence exists. If every vertex is fired once, it loses 2 chips from firing, and gains 1 from each of its 2 neighbors firing, for a net loss of 0.

Definition 2.4. We say configuration C_1 *leads to* configuration C_2 on a graph G if there is a valid sequence of moves that begins at C_1 and ends at C_2. In this case, we write $C_1 \to C_2$.

Problem 4: (10 points) Show that $C_1 \to C_2$ does not necessarily imply $C_2 \to C_1$.

Consider three vertices connected by an edge. Start with an initial configuration $(3, 0, 0)$. From problem 1, this configuration leads to $(1, 1, 1)$. From here, one fire leads to $(0, 2, 1)$ or one other case by symmetry. From here, there is only one fire, leading to $(1, 0, 2)$. From here, there is only one fire, leading us back to $(1, 1, 1)$. Thus, $(1, 1, 1)$ does not lead back to $(3, 0, 0)$.

Problem 5: (10 points) Given the graph below, either show that the C_1 leads to C_2, or prove that it does not.
$C_1 = (5, 0, 0, 0, 0, 4)$.
$C_2 = (2, 0, 2, 1, 0, 4)$.

Yes. Use firing sequence $(1, 1, 2, 3, 1, 2, 1, 3, 6, 4, 5, 2)$.

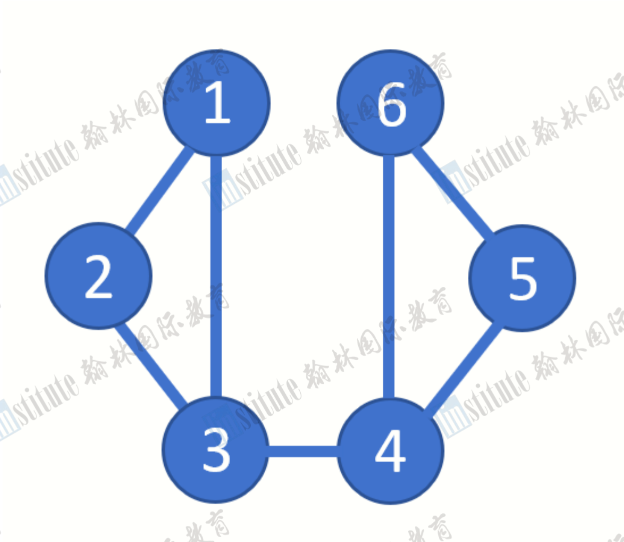

Figure 3: Graph for question 5.

Problem 6: (10 points) Given the two configurations below, draw a graph on 5 vertices for which one of the configurations leads to the other. Give the firing sequence that leads from one to the other.
$C_1 = (8, 0, 0, 0, 0)$.
$C_2 = (3, 3, 0, 1, 1)$.

There are many possible solutions here. A 4-star with two star points connected works well.

3 Finite and Infinite Games

Definition 3.1. A chip firing game with an initial configuration C is called *finite* if there exists a finite sequence of fires that results in a configuration with no valid fires. A chip firing game that is not finite is called *infinite*, ie. a game is called infinite if, after any sequence of moves, the game can continue.

Problem 7: (10 points) Prove that, in an infinite game, every vertex must be fired infinitely often. Note: You may assume there are a finite number of chips on the graph.

There is at least one vertex that is fired infinitely often. Each time it fires, its neighbor accumulates one chip. Since there are a finite number of chips, that neighbor must eventually fire to continue the game. Then it fires infinitely often as well. By this logic, all neighbors of all vertices fire infinitely often, meaning all vertices fire infinitely often.

Problem 8: (10 points) Prove that, after a finite game terminates, there must be at least one vertex in the graph that has never fired. *Hint: Try proving the contrapositive!*

We will prove that if every vertex fires, the game does not need to terminate. Consider the vertex that has not fired the longest in the graph. Since all of its neighbors have fired since it last fired, it must have accumulated a number of chips at least as large as the number of its neighbors. Thus, it is able to fire, and the game can always continue.

The following is our 'big theorem' for this test!

Problem 9: (20 points) On a graph with n vertices and m edges:

a Prove that if we start with more than $2m - n$ chips in any configuration, the game is infinite. *Hint: What is the total of the degrees for vertices in the graph?*

b For this part only, assume m is as large as it can be, ie. the graph has an edge between every pair of vertices. Prove that if we start with fewer than m chips, the game is finite. Note: This is true without the assumption, but the proof is a bit too complicated here.

c Prove that if the number of chips is between m and $2m-n$, there are initial configurations that result in both finite and infinite games.

a The degree total for such a graph is $2m$, which is a sum of n numbers. The total number of chips on the graph is at least $2m-n+1$, which is also a total of n numbers. Thus, at least one of the n chip total numbers must be equal to (or greater than) the corresponding degree numbers. Therefore, at least one vertex can always fire, and the game can always continue.

b Note that the total number of chips is less than $\frac{n(n-1)}{2}$. Since $(n-1)$ chips are required to fire, from the initial configuration, at most $\frac{n}{2}$ fires are possible. Perform all these fires. Each node gains a maximum of $\frac{n}{2}$ chips from this. As long as $n > 2$, this means no node has gained enough chips for an additional full fire. Since some nodes have fired, the total number of possible fires in the graph has now decreased. If we continue in this way, we necessarily decrease the number of fires to 0, so the game is finite. Note: If $n = 2$, we would be starting with 0 chips.

c The negative case is easy; simply start with a number of chips on each node equal to (or less than) one less than the degree of that node.
For the positive case, there are many ways to prove the result; this might be the cleanest. Do the following: Change each edge to an arrow pointing in either direction, and do it in such a way that there are no cycles (this is always possible). Place on each node a number of chips equal to the number of arrows leaving that node; this is possible since we have at least one chip per edge. Since there are no cycles in the graph, there must be at least one node whose arrows all point out. This node can be fired. Firing it is exactly equivalent to reversing the direction of all arrows touching

it. Reversing all such arrows also creates no cycles. As such, there is now a new node whose arrows point all out, so there is a new node that can be fired. This can continue forever, so the game is infinite.

Problem 10: (10 points) On the below graph, find a starting configuration with 6 chips that results in an infinite game.

Many possible solutions. One is to put all 6 chips on one of the vertices with 3 incident edges.

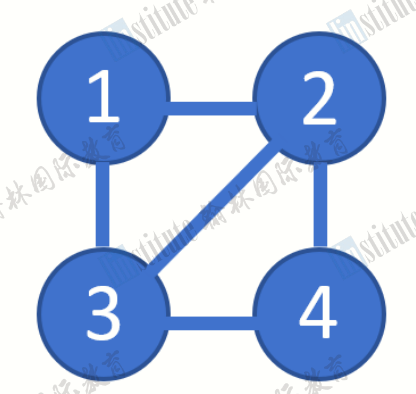

Figure 4: Graph for question 10.