Johns Hopkins Math Tournament 2021

Individual Round: Middle School

April 3rd, 2021

Instructions

- **Remember you must be proctored while taking the exam.**
- This test contains 45 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No outside help is allowed. This includes people, the internet, translators, books, notes, calculators, or any other computational aid. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor.
- Good luck!

1. Hanson swims at 2 miles per hour. How many miles can Hanson swim in five and a half hours?
2. What is the sum of the number of faces and edges of a triangular prism?
3. The circumference of a circle with area 9π can be expressed as $a\pi$. What is a?
4. How many integers x exist such that $x + \frac{1}{x}$ is also an integer?
5. How many factors of 32! are prime?
6. Metro trains pass through Rosslyn Station every 19 minutes. If Edward goes to Rosslyn Station at a random time, the probability that a train was at the station within 3 minutes of Edward's arrival time can be expressed in simplest form as $\frac{a}{b}$. What is $a + b$?
7. On Wednesday, Shams spent 8 hours metal detecting and sold his findings for \$124, while Jackie worked as a waitress from 10 AM to 4 PM and earned \$99. How much more did Jackie make per hour than Shams in cents?
8. J, H, M, and T are 4 distinct integers with values 1, 2, 3, and 4 in some order. What is the largest possible value of $JH \times MT$, where JH and MT are two-digit numbers formed by combining the digits of J&H and M&T respectively.
9. In a particular game, Cephen Sturry shoots 14 three-pointers, shown by the sequence 10110101100111, where 1's represent makes and 0's represent misses. In this game, if Sturry made a shot, the probability he also made his next shot can be expressed in simplest form as $\frac{a}{b}$. What is $a + b$?
10. Ranger is trying to get better at Chess, and decides that quantity is more important than quality. Thus, he plays 800 games each day, wins 53% of his games, ties 2% of his games, and loses the rest. How many days will it take until Ranger has at least 2021 more wins than losses?
11. What is the sum of the possible values of the units digit of the square of an integer?
12. Sergey has 8 different socks in his drawer, all of which are either red, blue, or yellow. In order to guess how many socks of each color are in the drawer, Kaden randomly draws pairs of socks with replacement. On his first draw, Kaden gets a red sock and a blue sock, and on his second draw, Kaden gets two blue socks. What is the difference between the maximum and minimum possible number of yellow socks in the drawer?
13. James wants to bake 314 pies for Pi Day (March 14th). One month beforehand, James bakes 1 pie on the first day, 2 pies on the second day, 3 pies on the third day, ..., 24 pies on the 24th day. After these 24 days, how many pies will James have left to bake?
14. If integer W has 2 prime factors and 8 total factors, how many factors does W^2 have?
15. In the city of Inondation, floods happen frequently. In any given 2-day period, the chance of a flood hitting the city at least once is 91%. If the chance of a flood hitting the city is the same for every day, what is the chance that a flood will hit Inondation on any single day as a percent?
16. In the same city of Inondation, the average amount of water that floods into the city in any 3-day period is 21 tons. What is the average amount of floodwater that one flood brings in tons? Note that you will need to use your answer to the previous problem to solve this one.
17. How many ways can the integers from 1 to 6 inclusive be ordered such that no product of two adjacent integers is odd?
18. Amanda uses a wheelbarrow to carry bottles of water. If she loads the wheelbarrow with 3 full bottles of water, its total weight is 23 pounds. If she loads the wheelbarrow with 1 full bottle of water and 2 empty bottles of water, its total weight is 17 pounds. Lastly, if she loads the wheelbarrow with 6 empty bottles of water, its total weight is 20 pounds. If the wheelbarrow and empty bottles of water both have positive weight, what is the sum of the weight of the wheelbarrow and a full bottle of water in pounds?

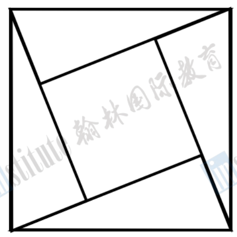
19. Suppose that the greatest area of a triangle with perimeter 26 and integer side lengths is A. What is A^2?

20. What is the largest integer less than 1000 that is not relatively prime with 2021? Recall that the prime factorization of 2021 is 43×47.

21. What is the area of the largest triangle whose vertices can be formed by the x and y-intercepts of the equation $y = -(x+3)(x+2)(x-6)$?

22. How many pairs of integers (b, c) exist such that b and c are both between 1 and 60 inclusive, and $b^2 - c^2$ is a positive integer that is divisible by 29?

23. In the figure below, the smaller square has side length 7 and the each of the four identical right triangles has a shorter leg of length 3. If the area of the larger square is X, what is X^2?



24. At the Krusty Krab, 15 cooks can make 15 Krabby Patties in 15 minutes. How long in minutes does it take 25 cooks to make 75 Krabby Patties?

25. Circle A has diameter 12. Circles B and C have diameters 7 and 5 respectively, and are inscribed inside A such that both are internally tangent to A and externally tangent to each other. The area of the region inside A but outside B and C can be expressed as $\frac{x}{y}\pi$ where $\frac{x}{y}$ is a fraction expressed in simplest form. What is $x + y$?

26. Consider the range of all real numbers between $-3\frac{3}{4}$ and $6\frac{1}{4}$ inclusive. If a real number n is randomly picked from this range, the probability that when rounded to the nearest integer, n is rounded to a multiple of 3 can be written in simplest form as $\frac{a}{b}$. What is $a + b$?

27. Morty drops a pickle into a vat of acid. Every minute, 25% of the remaining pickle is dissolved by the acid. In how many minutes will 25% of the original pickle by left? Round your answer to the nearest minute.

28. What is the largest distance between a circle of diameter 14 that is centered at $(2, 2)$ and a circle of diameter 4 that is centered at $(11, -10)$?

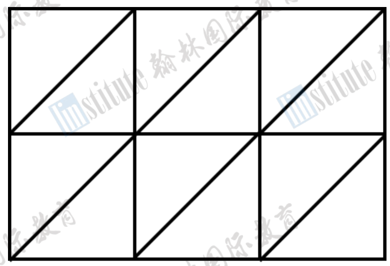
29. The Moonmen and the Martians are playing each other in a 3-inning baseball game. In each inning, there is a 40% chance the Moonmen score exactly 2 points, and a 75% chance the Martians score exactly 1 point. The probability that the game is tied can be expressed in simplest form as $\frac{a}{b}$. What is $a+b$? Note that the events of the Moonmen scoring and the Martians scoring have no impact on each other, and each inning both teams either score according to the point values given, or do not score at all.

30. Arthur's favorite numbers are 2, 4, and 42. He realizes that if he adds a real number a to all three of his favorite numbers, the resulting sequence $\{2 + a, 4 + a, 42 + a\}$ is a geometric sequence. If a can be expressed in simplest form as $\frac{b}{c}$ where $c > 0$, what is $b + c$?

31. If $a : b = 2 : 3$, $b : c = 4 : 5$, and $c : d = 3 : 7$, then the value of $\frac{b+c+d}{2a}$ can be expressed in simplest form as $\frac{x}{y}$. What is $x + y$?

32. Hamza has a standard deck of 52 cards, and shuffles the deck so that it is in a random order. If he then looks through the deck, the probability that the Ace of Diamonds appears before any face card can be expressed in simplest form as $\frac{a}{b}$. What is $a+b$? Note that the "face cards" are the Jacks, Queens, and Kings of any suite.

33. In the figure below, how many ways are there to get from the bottom left corner to the top right corner if paths must follow lines in the figure and are only allowed to go to upwards, rightwards, or diagonally up and to the right?



34. Using the same figure and definition of a path given in the previous problem, the average path length can be expressed in simplest radical form as $\frac{a+b\sqrt{2}}{c}$ for integers a, b, and c. What is $a+b+c$? Assume that the figure is composed of 6 unit squares.

35. Find k if $k = 30 + \sqrt{30 + \sqrt{30 + \sqrt{30 + ...}}}$

36. How many four digit multiples of 3 can be created from the digits 1, 2, and 3? Note that repeating digits is allowed.

37. How many positive integers less than $10,000$ satisfy the property that the products of their digits is 210?

38. What is the sum of the possible values of X such that the set $\{X, 57, 31, 43, 49\}$ has the same mean and median?

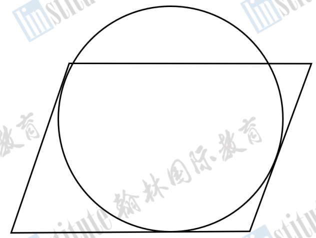
39. Ryan notices that at 12 pm, the hour hand and the minute hand of a clock are directly on top of each other, and thus form a 0 degree angle. The number of minutes Ryan will need to wait until the hour hand and the minute hand make a 99 degree angle for the second time can be expressed in simplest form as $\frac{a}{b}$. What is $a+b$?

40. In $\triangle ABC$, $AB = 22$ and $AC = 33$. The angle bisector of $\angle BAC$ intersects $\overline{BC}$ at X. Suppose that BX and CX are integers. What is the sum of all possible values of BC?

41. Douglas challenges Brian and Kenny to find the smallest positive integer that is twice a perfect square and three times a perfect cube. Brian finds the correct answer, but Kenny accidentally instead finds the smallest positive integer that is three times a perfect square and twice a perfect cube. What is the positive difference between Brian and Kenny's answers?

42. A triangle with area x has side lengths 3, 7, and x. What is the product of all possible values of x?

43. In the figure below, the parallelogram has altitudes of length 6 and 8, and the circle is tangent to 3 of the sides of the parallelogram. The area of the portion of the circle that is outside the parallelogram can be expressed in simplest radical form as $\frac{a\pi+b\sqrt{c}}{d}$ where a, b, c, and d are integers. What is $a+b+c+d$?



44. A trapezoid has perpendicular diagonals, one of which has length 12. The trapezoid's long base is length 10. The trapezoid's remaining diagonal and its top base are both length x. x can be expressed in simplest form as $\frac{a}{b}$. What is $a+b$?

45. Let X be the number of positive integers less than or equal to 2021^2 that are relatively prime to 2021^2. What is $X \mod 1000$? Recall that the prime factorization of 2021 is 43×47.

Middle School Individual Solutions

1. $\boxed{11}$. 2 mph $\times$ 5.5 hours = 11 miles.

2. $\boxed{14}$. A triangular prism has 5 faces, two are triangle bases, and three are the lateral quadrilaterals. A triangular prism then has 9 edges, 3 that make up each of the triangle bases, and 3 that connect the vertices of the triangle bases. Thus, the answer is $5 + 3 + 3 + 3 = 14$.

3. $\boxed{6}$. If the area of the circle is 9π, then the radius r can be expressed as $\pi r^2 = 9\pi$, and solving we find that $r = 3$. Thus, the circumference of the circle is $2\pi r = 6\pi$, so the answer is 6.

4. $\boxed{2}$. Since x is always an integer, we only need to find an x such that $\frac{1}{x}$ is also an integer. First, notice that if $x = 1$ or $x = -1$, $\frac{1}{x}$ is 1 or -1, and thus is clearly an integer. We also see that if $|x| > 1$, $0 < |\frac{1}{x}| < 1$. Thus 1 and -1 are the only integers that make $x + \frac{1}{x}$ also an integer. So the answer is 2.

5. $\boxed{11}$. Since $32! = 32 \times 31 \times ... \times 2 \times 1$, we see that the prime factors of 32 are exactly the prime numbers between 32 and 1. Thus, we find that the numbers 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, and 31 are prime, and counting these up we find 11 prime factors.

6. $\boxed{25}$. Consider an arbitrary interval of 19 minutes. Assume that the train passes the station at $t = 0$ and thus also at $t = 19$. In this specific interval, Edward will be arrive within 3 minutes of the train if he arrives between $t = 0$ and $t = 3$, or within $t = 16$ and $t = 19$, for a probability of $\frac{6}{19}$. Now, notice that even though for this example we assumed that the train arrived at $t = 0$ and $t = 19$, the same resulting probability would be found regardless of what arrival time we chose for the train.

   For example, if instead the train arrived at $t = 2$, then it would also arrive at $t = 21$ (notice that even though this is outside our 19 minute interval, it still affects what times Edward can arrive!). Thus, Edward can now arrive between $t = 0$ and $t = 5$, or between $t = 18$ and $t = 19$, so the probability is still $\frac{6}{19}$.

7. $\boxed{100}$. Shams' hourly income is $\dfrac{\$124}{8 \text{ hours}} = \$15.50$. Since Jackie works for 6 hours, her hourly income is $\dfrac{\$99}{6 \text{ hours}} = \$16.50$. Thus, Jackie earns \$1 more dollar each hour, or 100 cents.

8. $\boxed{1312}$. We want to place the two largest digits in the tens place of JH and MT. Thus, the largest product will either be the result of 41×32 or of 42×31. If we try both, we find that 41×32 is larger, thus the answer is $41 \times 32 = 1312$.

   Note that if we do not want to try both possibilities, we can also immediately determine that 41×32 will be larger because $(41, 32)$ and $(42, 31)$ have the same sum, but $(41, 32)$ are closer together. Algebraically, this is because for numbers x, y with $x > y$, $(x + 1)(y - 1) = xy + y - x - 1$, and $y - x - 1$ will be negative since $x < y$, and thus $(x + 1)(y - 1) < xy$.

9. $\boxed{3}$. In the sequence of 14 numbers, we find every instance in which a 1 is followed by another 1, and divide that count by the number of instances in which a 1 is followed by anything. Thus, the resulting fraction is $\frac{4}{8} = \frac{1}{2}$. Note that we do not count the last 1 in the sequence in the denominator because there is no number after it.

10. $\boxed{32}$. Since each day Ranger wins 53% of his games and loses $100\% - 53\% - 2\% = 45\%$ of his games, after a day he will have $800 \times (53\% - 45\%) = 64$ more wins than loses. Thus, the answer will be $\lceil \frac{2021}{64} \rceil = 32$, where $\lceil x \rceil$ denotes the ceiling function, or the least integer greater than or equal to x.

11. $\boxed{25}$. Note that the units digit of the square of an integer x will only depend on the units digit of x. Thus, we square all the integers from 0 to 9 inclusive and find that the possible units digits of a square are 0, 1, 4, 9, 6, and 5.

12. $\boxed{5}$. The maximum number of yellow socks is found when all the socks that Kaden did not see are yellow. Since Kaden saw at least 1 red sock and at least 2 blue socks, there can be up to $8-(1+2)=5$ yellow socks. On the other hand, the minimum number of yellow socks is found when none of the socks are yellow, or 0.

13. $\boxed{14}$. To find the sum 1+2+...+23+24, we can pair up all the integers, i.e. $(1,24),(2,23),(3,22),...,(12,12)$, and find that all these pairs sum to 25. Since there are 12 such pairs, the sum is $25\times 12=300$. Thus, James will have $314-300=14$ pies left to bake at the end.

14. $\boxed{21}$. Suppose the two prime factors of W are A and B. Thus, $W=A^a\times B^b$ where a and b are integers at least 1. Thus, the number of factors of W is $(a+1)(b+1)=8$, meaning that either $a=1$ and $b=3$ or $a=3$ and $b=1$. Next, if we look at W^2, we find that $W^2=A^{2a}\times B^{2b}$, a number that has $(2a+1)(2b+1)$ factors. Thus, for both possible values of (a,b), W^2 will have $(2\times 1+1)(2\times 3+1)=21$ factors.

15. $\boxed{70}$. The chance of no floods hitting the city in a 2-day period is $1-0.91=0.09$. Let's call the chance of a flood NOT hitting the city on any given day q. Since the chance of a flood hitting (or not hitting) the city is the same for each day, then $q^2=0.09$. Thus, $q=0.3$. The chance that a flood will hit Inondation on any given day is thus $1-0.3=\boxed{0.7}$.

16. $\boxed{10}$. Recall from the previous question that the chance of a flood hitting the city on any given day is 0.7. Since a 3-day period brings on average 21 tons, and since the chance of a flood occurring on each day is the same, then each day should bring an average of 7 tons of floodwater. The average amount of floodwater on a day = average amount of floodwater per flood × chance of a flood. Thus, if average amount of floodwater per flood $=x$, then $7=x\times 0.7$. $x=\boxed{10}$.

17. $\boxed{144}$. Note that the product of two integers is only odd if both integers are odd. Thus, this question is equivalent to finding the number of ways to order the integers from 1 to 6 inclusive such that no two odd numbers are next to each other. Let E denote an even number and O denote an odd number. We find that there are 4 general configurations of even and odd numbers that satisfy the requirement of no adjacent odd numbers:

- $EOEOEO$
- $OEOEOE$
- $OEOEEO$
- $OEEOEO$

Finally, in each of these general configurations, there are 3!3! ways of filling in the three E's and three O's with even and odd numbers, thus the answer is $4\times 3!3!=144$.

18. $\boxed{13}$. Let W be the weight of an empty wheelbarrow, F be the weight of full water bottle, and E be the weight of an empty water bottle. Thus, from the information given, we can set up the following equations:

$$W+3F=23$$
$$W+1F+2E=17$$
$$W+6E=20$$

If we subtract the first equation from three times the second equation, we get:

$$2W+6E=28$$

If we then subtract the third equation from this one, we find that $W=8$. Now by substituting W back into the first and third equations, we can quickly find that $F=5$ and $E=2$. Thus, the answer is $W+F=13$.

19. $\boxed{1040}$. In general, for a given perimeter, shapes have their area maximized by setting all side lengths to be equal. In this problem, we cannot exactly do this because the side lengths must also be integers. However, the closest we can get to making all side lengths equal is having the three side lengths be 8, 9, and 9. If 8 is the base of this isosceles triangle, the height is $\sqrt{9^2 - 4^2} = \sqrt{65}$. Thus, A is $4\sqrt{65}$, and $A^2 = 16 \times 65 = 1040$.

20. $\boxed{989}$. Since the only two prime factors of 2021 are 43 and 47, numbers must be a multiple of 43 and 47 to *not be* relatively prime with 2021. We find that the largest multiple of 43 that is less than 1000 is 989, and the largest multiple of 47 that is less than 1000 is 987. Thus, the answer is 989 since it is larger.

21. $\boxed{162}$. From the equation, we can immediately tell that the three x-intercepts are $x = -3$, $x = -2$ and $x = 6$ since these are the values of x that make y equal to 0. Now, we can find the y-intercept by plugging in $x = 0$ into the equation, i.e. the y-intercept is $-(0+3)(0+2)(0-6) = 36$.

    In order to create a triangle with positive area, we must choose the y-intercept as one of the vertices, because otherwise all three vertices of the triangle will lie on the line $y = 0$. From this point, we want to make the base of the triangle as long as possible in order to maximize the area, so we will choose $(-3, 0)$ and $(6, 0)$ as the other two vertices of the triangle. Since the resulting triangle has a base length of 9 and a height of 36, it's area is 162.

22. $\boxed{93}$. Since $b^2 - c^2 = (b+c)(b-c)$, we will consider all the pairs (b, c) that either add up to a multiple of 29 or whose difference is a multiple of 29. Also, keep in mind that $b > c$ since the problem requires that $b^2 - c^2$ is positive. Thus, let us first consider the pairs that add up to a multiple of 29:

    - $b + c = 29$. Thus, the possible pairs are $(28, 1), (27, 2), ..., (15, 14)$, for a total of 14 possible pairs.
    - $b + c = 58$. Thus, the possible pairs are $(57, 1), (56, 2), ..., (30, 28)$, for a total of 28 possible pairs.
    - $b + c = 87$. Thus, the possible pairs are $(60, 27), (59, 28), ..., (44, 43)$, for a total of 17 possible pairs.
    - $b + c = 116$. Thus, the possible pairs are $(60, 56), (59, 57)$, for a total of 2 possible pairs. Note that no greater sum of $b + c$ is possible since b and c must be less than or equal to 60.

    Now, we will consider the pairs whose difference is a multiple of 29:

    (a) $b - c = 29$. Thus, the possible pairs are $(30, 1), (31, 2), ..., (60, 31)$, for a total of 31 possible pairs.

    (b) $b - c = 58$. Thus, the possible pairs are $(59, 1), (60, 2)$ for a total of 2 possible pairs. Note that no greater difference of $b - c$ is possible since the maximum possible difference is $60 - 1 = 59$.

    Thus, if we sum all the possible pairs found above, we get $14 + 28 + 17 + 2 + 31 + 2 = 94$. However, note that we over count by 1 here because the pair $(58, 29)$ both sums to a multiple of 29 and has a difference that is a multiple of 29. Thus the answer is $94 - 1 = 93$.

23. $\boxed{11881}$. First, note that the four triangles must be right triangles because we are given that the inner shape is a square. Furthermore, note that the longer leg of the triangles has length $7 + 3 = 10$. Thus, each of the triangles has area $\frac{10 \times 3}{2} = 15$, and the smaller square has area 49. Thus, the larger square has area 109 and the answer is $109^2 = 11881$.

24. $\boxed{45}$. Imagine that each of the 15 cooks make one of the 15 Krabby Patties. Then it's obvious that one cook makes one Krabby Patty in 15 minutes (all the cooks would finish at the same time). If there are 25 cooks trying to make 25 Krabby Patties, then they can once again allocate one Krabby Patty to each cook. Since $25 \times 3 = 75$, the cooks will finish in $15 \times 3 = 45$ minutes.

25. $\boxed{37}$. Notice that circles B and C are completely inside circle A because they are internally tangent. Furthermore, circles B and C do not overlap because they are externally tangent to each other. Thus, since circle A has area 36π, circle B has area $\frac{49\pi}{4}$, and circle C has area $\frac{25\pi}{4}$, the area inside of circle A but outside B and C is $36\pi - (\frac{49\pi}{4} + \frac{25\pi}{4}) = \frac{35\pi}{2}$, thus the answer is 37.

26. $\boxed{11}$. n will be rounded to a multiple of 3 if it lands in one of the following ranges:

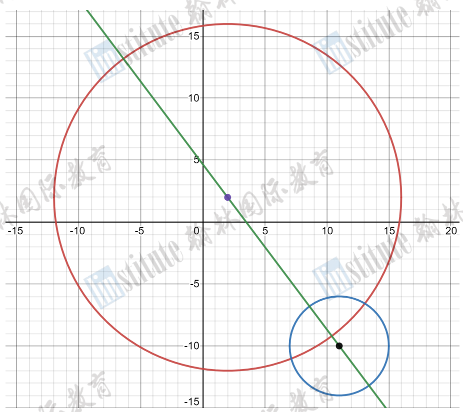
- $n \in [-3\frac{1}{2}, -2\frac{1}{2}]$
- $n \in [-\frac{1}{2}, \frac{1}{2}]$
- $n \in [2\frac{1}{2}, 3\frac{1}{2}]$
- $n \in [5\frac{1}{2}, 6\frac{1}{4}]$

The sum of the lengths of these intervals is $1 + 1 + 1 + \frac{3}{4} = \frac{15}{4}$. The total length of the interval is 10, thus the answer is $\frac{15}{40} = \frac{3}{8}$.

27. $\boxed{5}$. While we could set up the equation $(\frac{3}{4})^n = \frac{1}{4}$ and try solving for n to get an exact answer before rounding, doing so isn't easy without a calculator, so instead we will go minute by minute:

- After 1 minute, $\frac{3}{4}$ of the original pickle will be left.
- After 2 minutes, $(\frac{3}{4})^2 = \frac{9}{16}$ of the original pickle will be left.
- After 3 minutes, $(\frac{3}{4})^3 = \frac{27}{64}$ of the original pickle will be left.
- After 4 minutes, $(\frac{3}{4})^4 = \frac{81}{256}$ of the original pickle will be left. Notice that this fraction is approximately $\frac{1}{3}$, so we still haven't gotten past the desired $\frac{1}{4}$ proportion.
- After 5 minutes, $(\frac{3}{4})^5 = \frac{243}{1024}$ of the original pickle will be left. Notice that this fraction is smaller than $\frac{250}{1000} = \frac{1}{4}$ since the numerator is smaller than 250 and the denominator is greater than 1000. Since this fraction is closer to $\frac{1}{4}$ than $\frac{81}{256}$, or the fraction 1 minute ago, if we round, the answer will be 5 minutes.

28. $\boxed{24}$. Refer to the figure below for clarity. Notice that the centers of the two circles (denoted by the purple and black dot in the figure), are $\sqrt{(11-2)^2 + (-10-2)} = 15$ units apart. Now, in order to maximize the distance between two points, one on each circle, we want to place each point on the line that connects the two centers. One point will be on the top left of the red circle (where the red circle intersects the green line), and the other point will be on the bottom right of the blue circle (where the blue circle intersects the green line). The distance between these two points is the sum of the distance between the two centers and the radius of both circles, or $15 + 7 + 2 = 24$.



29. $\boxed{1897}$. Since the Martians can only score 0, 1, 2, or 3 points, and the Moonmen can only score 0, 2, 4, or 6 points, the game can only be tied either $0-0$ or $2-2$.

First, consider the case where both teams score 0 points. In other words, both teams fail to score in all 3 innings. For the Moonmen, this probability is $(1-\frac{2}{5})^3 = \frac{27}{125}$. For the Martians, this probability is $(1-\frac{3}{4})^3 = \frac{1}{64}$. We will multiply these two probabilities in order to find the overall probability that the game ties $0-0$.

Next, consider the case where both teams score 2 points. For the Moonmen, this happens if they score 2 points in exactly 1 inning, and fail to score in the other 2 innings. Since there are $\binom{3}{1} = 3$ ways this can occur, the probability is $3 \times (\frac{2}{5}) \times (1-\frac{2}{5})^2 = \frac{54}{125}$. For the Martians, this happens if they score 1 point in exactly 2 innings, and fail to score in the other 1 inning. Since there are $\binom{3}{2} = 3$ ways this can occur, the probability is $3 \times (\frac{3}{4})^2 \times (1-\frac{3}{4}) = \frac{27}{64}$.

The total probability is thus the sum of the two cases, or

$$\frac{27}{125} \times \frac{1}{64} + \frac{54}{125} \times \frac{27}{64} = \frac{27 + 27 \times 54}{64 \times 125} = \frac{27 \times 5 \times 11}{64 \times 125} = \frac{27 \times 11}{64 \times 25} = \frac{297}{1600}$$

.

30. $\boxed{-8}$. Since $\{2+a, 4+a, 42+a\}$ is a geometric sequence, we have:

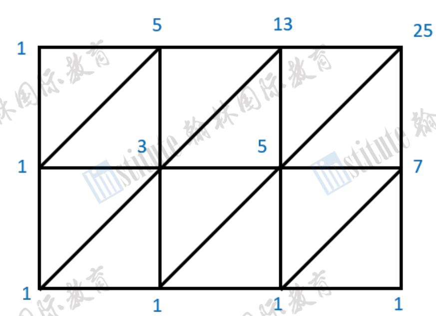
$$\frac{2+a}{4+a} = \frac{4+a}{42+a}$$
$$(4+a)^2 = (2+a)(42+a)$$
$$a^2 + 8a + 16 = a^2 + 44a + 84$$
$$36a = -68$$
$$a = -\frac{68}{36} = -\frac{17}{9}$$

Since the problem specifies that $c > 0$, the final answer is $-17 + 9 = -8$.

31. $\boxed{39}$. Suppose $b = 12$ (the LCM of 3 and 4), and $c = 15$ (the LCM of 5 and 3). Then, we see that $a = 8$, $d = 35$ satisfies all four ratios given in the problem. Thus, the answer is $\frac{12+15+35}{16} = \frac{62}{16} = \frac{31}{8}$.

32. $\boxed{14}$. This problem isn't as complicated as it may first seem. Despite there being 52 cards in the deck, the only cards we care about are the 12 face cards (3 face cards, 4 suites), and the Ace of Diamonds. If we just isolate the ordering of these 13 cards in the deck, then the problem reduces down to the probability that the Ace of Diamonds is the first of the 13 cards. Since each of the 13 cards has the same chance of being the first, this probability is just $\frac{1}{13}$.

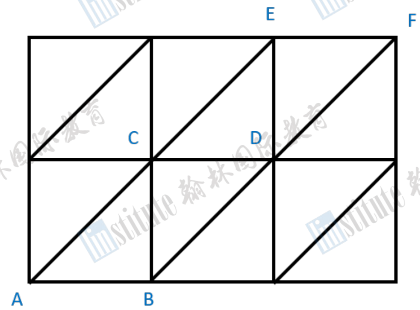
33. $\boxed{25}$. One systematic way of counting the total number of paths is to count the number of paths to every vertex in the figure. Using this method, note that each vertex V's number of paths count is the sum of the number of paths to the vertex left of V, below V, and diagonally bottom-left of V. Thus, we end up with the following counts on the vertices in the figure:



34. $\boxed{132}$. First, observe that every path either uses 0 diagonal path segments, 1 diagonal path segment, or 2 diagonal path segments, and that paths of each category have different length. Thus, we will count the number of each path type:

- The number of paths with 0 diagonal path segments can be immediately found by $\binom{5}{2} = 10$. Note that these paths all have length 5, and thus the total length of the 10 paths is 50.
- There are 3 paths with 2 diagonal path segments. Using the figure below, these 3 paths follow the vertices $ACEF$, $ABDF$, and $ACDF$. Note that these paths all have length $2\sqrt{2}+1$, and thus the total length of the 3 paths is $6\sqrt{2}+3$.

- Since in the previous problem we found that there were 25 total paths, it must be the case that the remaining 12 paths have 1 diagonal path segment. Note that these paths all have length $\sqrt{2}+3$, and thus the total length of the 12 paths is $12\sqrt{2}+36$.



Thus, the average path length is the total length of all 25 paths divided by 25, or:

$$\frac{50+6\sqrt{2}+3+12\sqrt{2}+36}{25}=\frac{89+18\sqrt{2}}{25}$$

35. $\boxed{36}$. First, we will find that value of $\sqrt{30+\sqrt{30+\sqrt{30+\ldots}}}$. Suppose this quantity has value x. Thus, we see that $x^2=30+x$. Solving this quadratic, we find that $(x-6)(x+5)=30$, and thus x can either be 6 or -5. However, remember that the square root is a strictly positive function, and thus since the original expression is in terms of square roots, $x=6$ is the only acceptable value of this term. Thus, the answer is $30+6=36$.

36. $\boxed{27}$. Note that an integer is a multiple of 3 if the sum of its digits is a multiple of 3. Thus, we can count the ways that four digits, each either 1, 2, or 3, sum to a multiple of 3:

- 1122. Note that there are $\frac{4!}{2!2!}=6$ ways of ordering these four numbers.
- 1113. Note that there are $\frac{4!}{3!}=4$ ways of ordering these four numbers.
- 2223. Note that there are $\frac{4!}{3!}=4$ ways of ordering these four numbers.
- 1233. Note that there are $\frac{4!}{2!}=12$ ways of ordering these four numbers.
- 3333. Note that there is 1 way of ordering these four numbers.

Thus, the answer is the sum obtained from the cases above, or $6+4+4+12+1=27$.

37. $\boxed{54}$. First, we find that the prime factorization of 210 is $2\times3\times5\times7$. From here, let's split into cases based on number of digits. First, note that since the problem specifies integers less than $10,000$, we are only considering integers with 4 or less digits. Furthermore, also notice that no two-digit integers satisfy this requirement, as even the product of the digits of 99 do not reach 210.

(a) Thus, the first case is integers with 4 digits. From here, we have several sub-cases.
   - First, we can use all 4 prime factors, $\{2,3,5,7\}$ as our digits. We can create $4!=24$ valid numbers from these 4 digits.
   - We can also combine $2\times3=6$ and use $\{1,5,6,7\}$ as digits. Again, there are $4!=24$ valid numbers from these 4 digits.

(b) Now, the second case is integers with 3 digits. The only possible 3 digits are $\{5,6,7\}$, which can be ordered $3!=6$ ways.

Thus, the answer is $24+24+6=54$.

38. $\boxed{145}$. First, observe that since the set contains an odd number of elements, the median of the set must be exactly one of the members. Furthermore, regardless of the value of X, 57 cannot be the median because at most only one element in the set (X) can be greater than it. Likewise, 31 cannot be the median because at most one element in the set (X) can be less than it. Thus, we will consider three cases:

- In the first case, X is the median and the mean. If X is the mean, then $\frac{57+31+43+49+X}{5} = X$. Solving, we find that $X = 45$. Since this value of X also makes X the median, $X = 45$ is one valid solution.
- In the second case, 43 is the median and the mean. If 43 is the mean, then $\frac{57+31+43+49+X}{5} = 43$. Solving, we find that $X = 35$. Since this value of X also makes 43 the median, $X = 35$ is another valid solution.
- In the third case, 49 is the median and the mean. If 49 is the mean, then $\frac{57+31+43+49+X}{5} = 49$. Solving, we find that $X = 65$. Since this value of X also makes 49 the median, $X = 65$ is another valid solution.

Thus, the sum of the possible values of X is $45 + 35 + 65 = 145$.

39. $\boxed{533}$. Before we directly approach the problem, consider what happens to the relation between the hour hand and the minute hand each minute. Every minute, the minute hand moves 1 minute, and since there are 60 minutes marked on a clock, each minute the minute hand moves $\frac{360}{60} = 6°$. As for the hour hand, every minute it moves $\frac{1}{60}^{\text{th}}$ of an hour. Since there are 12 hours marked on a clock, each hour on the clock takes up $\frac{360}{12} = 30°$ degrees. Thus, $\frac{1}{60}^{\text{th}}$ of an hour takes up $\frac{30}{60} = 0.5°$. Thus, every minute, the minute hand moves 5.5° further than the hour hand.

Given this, we will now approach the problem. Since we start at 12 pm, there are initially 0° between the hour hand and the minute hand. Since we found each minute the minute hand moves 5.5° further than the hour hand, it will take $\frac{99}{5.5} = 18$ minutes for the angle between the two hands to be 99° for the first time. However, the problem is asking for the *second* time that the measure between the hands is 99°. This happens when the two hands are $360° - 99° = 261°$ apart. Thus, the answer is $\frac{261}{5.5} = \frac{522}{11}$.

40. $\boxed{260}$. Let $BX = a$ and $XC = b$. Thus, from the angle bisector theorem, we know that $\frac{22}{a} = \frac{33}{b}$. Furthermore, since a and b must be integers, a must be a positive even number so that $b = \frac{3}{2}a$ is also an integer. Just from this ratio, we know that values of (a, b) to consider are $(2, 3), (4, 6), ...,$ etc.

Next, from the triangle inequality, we also notice that $a+b$ must be greater than $33-22 = 11$, and less than $33+22 = 55$. Thus, we can conclude that the possible values of (a, b) are $(6, 9), (8, 12), ..., (20, 30)$. Thus, the possible values of the length of BC are 15, 20, ..., 50, and the sum of these numbers is 260.

41. $\boxed{216}$. First, let us find Brian's answer (the correct answer). Thus, we are looking for x such that x is the smallest integer that is twice a perfect square and three times a perfect cube. Since x is both twice a number and three times a number, x must contain 2 and 3 in its prime factorization. In order to make x as small as possible, suppose that the only prime factors of x are 2 and 3. Thus, $x = 2^a3^b$ for positive integers a and b. Since x is twice a perfect square, a must be odd and b must be even. Since x is three times a perfect cube, b must be 1 more than a multiple of 3 (i.e. $b \equiv 1 \mod 3$), and a must be divisible by 3. To summarize:

$$a \equiv 1 \mod 2$$
$$b \equiv 0 \mod 2$$
$$a \equiv 0 \mod 3$$
$$b \equiv 1 \mod 3$$

We find that the smallest values of a and b that satisfy these conditions are $a = 3$ and $b = 4$. Thus, Brian's number is $2^3 \times 3^4 = 648$.

Next, we will repeat the procedure above to find Kenny's number y. Again, let $y = 2^c 3^d$. Since y is three times a perfect square and twice a perfect cube, we have the following conditions:

$$c \equiv 0 \mod 2$$
$$d \equiv 1 \mod 2$$
$$c \equiv 1 \mod 3$$
$$d \equiv 0 \mod 3$$

We find that the smallest values of c and d that satisfy these conditions are $c = 4$ and $d = 3$. Thus, Brian's number is $2^4 \times 3^3 = 432$. Thus, the answer is $648 - 432 = 216$.

42. $\boxed{40}$. Solving this problem requires Heron's Formula, which states that for a triangle with side lengths a, b, and c, the area of the triangle is:

$$\sqrt{s(s-a)(s-b)(s-c)}$$

where s is the semi-perimeter of the triangle, or $\frac{a+b+c}{2}$.

Thus, applying what we know from the problem, we find that $s = \frac{3+7+x}{2}$, and:

$$x = \sqrt{(\frac{10+x}{2})(\frac{10+x}{2} - 3)(\frac{10+x}{2} - 7)(\frac{10+x}{2} - x)}$$

$$x^2 = (\frac{10+x}{2})(\frac{10+x-6}{2})(\frac{10+x-14}{2})(\frac{10+x-2x}{2})$$

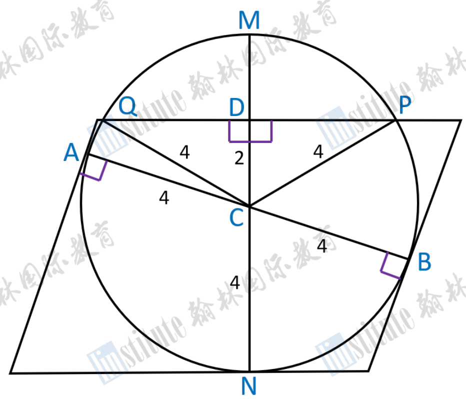
$$16x^2 = (10+x)(x+4)(x-4)(10-x)$$
$$16x^2 = (100-x^2)(x^2-16)$$
$$16x^2 = -x^4 + 116x^2 - 1600$$
$$x^4 - 100x^2 + 1600 = 0$$
$$(x^2-80)(x^2-20) = 0$$

Thus, since x must be positive as it is a side length of a triangle, $x = \sqrt{20}$ or $x = \sqrt{80}$. Note that by the triangle inequality, x must be at least $7 - 3 = 4$, and x must be at most $7 + 3 = 10$. Thus, both these values of x are possible side lengths because $\sqrt{20} > \sqrt{16} = 4$ and $\sqrt{80} < \sqrt{81} = 9$. Thus, the answer is $\sqrt{20}\sqrt{80} = \sqrt{1600} = 40$.

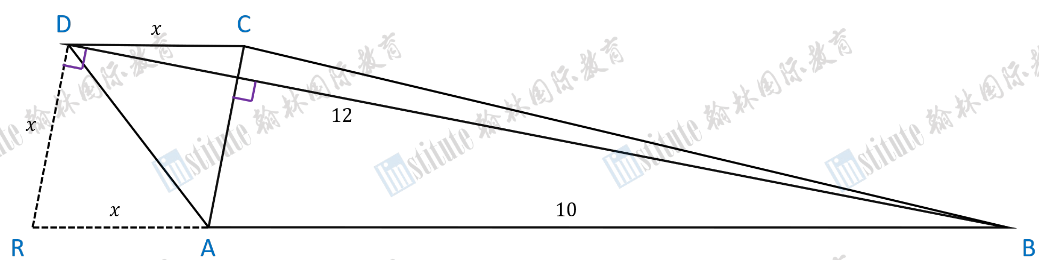
43. $\boxed{10}$. Refer to the figure below for clarity. First, draw the trapezoid altitude $\overline{AB}$. Since the circle is tangent to 3 sides of the trapezoid, $\overline{AB}$ is also a diameter of the circle. Moreover, $\overline{AB}$ must be the altitude of length 8 because the other altitude, $\overline{DN}$, is shorter than the diameter of the circle.

Let C be the center of the circle. Next, draw radii $\overline{QC}$ and $\overline{PC}$, where Q and P are the points where the circle intersects the top side of the trapezoid. Both these radii have length 4. Next, since $\overline{DN}$ is the other altitude of the trapezoid, $DN = 6$, and $DC = 2$. Moreover, note that $\angle QDC$ and $\angle PDC$ must be right angles since $\overline{DN}$ is an altitude. Thus, we find that $\triangle QDC$ and $\triangle PDC$ are right triangles with hypotenuse length twice a leg length, which implies that $\triangle QDC$ and $\triangle PDC$ are $30° - 60° - 90°$ triangles. Thus, $\angle QCP = 120°$.

As a result, we find that the area of the sector of the circle defined by arc QMP is $16\pi \times \frac{120°}{360°} = \frac{16\pi}{3}$, and the area of $\triangle QPC$ is $4\sqrt{3}$. Thus, the area outside the trapezoid is $\frac{16\pi}{3} - 4\sqrt{3} = \frac{16\pi - 12\sqrt{3}}{3}$, and the answer is $16 - 12 + 3 + 3 = 10$.



44. $\boxed{16}$. Refer to the figure below for clarity. Let our trapezoid be trapezoid $ABCD$, where $\overline{AB}$ is the long base of length 10 and $\overline{CD}$ is the short base. Let's say line segment $\overline{DB}$ is the diagonal equal to 12. $CD = AC = x$. Draw a line parallel to $\overline{AC}$ that goes through point D, and call the point where it intersects the extended line $\overline{AB}$ point R. Triangle $\triangle RDB$ is a right triangle, where $RB = RA + AB = x + 10$. $RD = x$, while $DB = 12$. Thus, using the Pythagorean theorem, $x^2 + 12^2 = (x + 10)^2$. Solving for x gives $x = \frac{11}{5}$. Thus, the answer is 16.



45. $\boxed{572}$. Before we directly approach the problem, let us make a few general observations about relatively prime numbers. Thus, let p be a prime number. Since p is prime, all the integers from 1 to $p - 1$ are relatively prime to p, and thus there are a total of $p - 1$ positive integers less than or equal to p that are relatively prime to p.

Next, consider p^2, where again p is prime. Since p^2 only has p as a factor, the integers:

$$\{1 \times p, 2 \times p, ..., (p - 1) \times p, p \times p\}$$

will be the only positive integers less than or equal to p^2 that are *not* relatively prime to p^2. Thus, there are $p^2 - p$ positive integers less than or equal to p^2 that are relatively prime to p^2.

Finally, consider p^2q^2 where p and q are both prime numbers. Since the only prime factors of these number are p and q, the integers:

$$\{1 \times p, 2 \times p, ..., (q^2p - 1) \times p, q^2p \times p\} \text{ AND } \{1 \times q, 2 \times q, ..., (p^2q - 1) \times q, p^2q \times q\}$$

will be the only positive integers less than or equal to p^2q^2 that are *not* relatively prime to p^2q^2. These two sets have size q^2p and p^2q respectively, but also notice that the integers:

$$\{1 \times pq, 2 \times pq, ..., (pq - 1) \times pq, pq \times pq\}$$

will appear in both sets, and thus will be double subtracted. Thus, accounting for this, the total number of integers less than or equal to p^2q^2 that are relatively prime to p^2q^2 will be:

$$p^2q^2 - q^2p - p^2q + pq = (p^2 - p)(q^2 - q).$$

Finally, we plug in $p = 43$ and $q = 47$ into the findings above to conclude that the number of positive integers less than or equal to 2021^2 that are relatively prime to 2021^2 is:

$$(43^2 - 43)(47^2 - 47) = 3904572.$$