Johns Hopkins Math Tournament 2021

Individual Round: Calculus

April 3rd, 2021

Instructions

- **Remember you must be proctored while taking the exam.**
- This test contains 10 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No outside help is allowed. This includes people, the internet, translators, books, notes, calculators, or any other computational aid. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor.
- Good luck!

1. The value of x in the interval $[0, 2\pi]$ that minimizes the value of $x + 2\cos x$ can be written in the form $a\pi/b$, where a and b are relatively prime positive integers. Compute $a + b$.

2. Compute the smallest positive integer n such that $\int_0^n \lfloor x \rfloor \, dx$ is at least 2021.

3. There is a unique ordered triple of real numbers (a, b, c) that makes the piecewise function $f(x) = \begin{cases} (x-a)^2 + b & \text{if } x \geq c \\ x^3 - x & \text{if } x < c \end{cases}$ twice continuously differentiable for all real x. The value of $a + b + c$ can be expressed as a common fraction p/q. Compute $p + q$.

4. There is a unique differentiable function f from $\mathbb{R}$ to $\mathbb{R}$ satisfying $f(x) + (f(x))^3 = x + x^7$ for all real x. The derivative of $f(x)$ at $x = 2$ can be expressed as a common fraction a/b. Compute $a + b$.

5. For real numbers x, let T_x be the triangle with vertices $(5, 5^3)$, $(8, 8^3)$, and (x, x^3) in $\mathbb{R}^2$. Over all x in the interval $[5, 8]$, the area of the triangle T_x is maximized at $x = \sqrt{n}$, for some positive integer n. Compute n.

6. Let f be a function whose domain is $[1, 20]$ and whose range is a subset of $[-100, 100]$. Suppose $\frac{f(x)}{y} - \frac{f(y)}{x} \leq (x - y)^2$ for all x and y in $[1, 20]$. Compute the largest value of $f(x) - f(y)$ over all such functions f and all x and y in the domain $[1, 20]$.

7. In three-dimensional space, let $\mathcal{S}$ be the surface consisting of all points $(x, y, 0)$ satisfying $x^2 + 1 \leq y \leq 2$, and let A be the point $(0, 0, 900)$. Compute the volume of the solid obtained by taking the union of all line segments with endpoints in $\mathcal{S} \cup \{A\}$.

8. Find the unique integer $a > 1$ that satisfies
$$\int_a^{a^2} \left(\frac{1}{\ln x} - \frac{2}{(\ln x)^3} \right) dx = \frac{a}{\ln a}.$$

9. Define a sequence $\{a_n\}_{n=0}^{\infty}$ by $a_0 = 1$, $a_1 = 8$, and $a_n = 2a_{n-1} + a_{n-2}$ for $n \geq 2$. The infinite sum
$$\sum_{n=1}^{\infty} \int_0^{2021\pi/14} \sin(a_{n-1}x)\sin(a_n x)\, dx$$
can be expressed as a common fraction p/q. Compute $p + q$.

10. A polynomial $P(x)$ of some degree d satisfies $P(n) = n^3 + 10n^2 - 12$ and $P'(n) = 3n^2 + 20n - 1$ for $n = -2, -1, 0, 1, 2$. Also, P has d distinct (not necessarily real) roots $r_1, r_2, \ldots, r_d$. The value of
$$\sum_{k=1}^{d} \frac{1}{4 - r_k^2}$$
can be expressed as a common fraction p/q. What is the value of $p + q$?

Calculus Solutions

1. Let $f(x) = x + 2\cos x$. The minimizer of $f(x)$ over $[0, 2\pi]$ is either 0, 2π, or some $x \in [0, 2\pi]$ satisfying $f'(x) = 0$. This last equation yields $1 - 2\sin x = 0$, which is equivalent to $\sin x = 1/2$, whose only solutions in $[0, 2\pi]$ are $x = \pi/6$ and $x = 5\pi/6$. Thus, the desired minimizer is either 0, $\pi/6$, $5\pi/6$, or 2π, and quickly plugging each of these values into the function f reveals that $f(5\pi/6)$ is the smallest, meaning that $a = 5$ and $b = 6$, so $a + b = \boxed{11}$.

2. In general, we have
$$\int_0^n \lfloor x \rfloor \, dx = \sum_{k=0}^{n-1} k = \frac{(n-1)n}{2},$$
so we seek the smallest positive integer n satisfying $\frac{(n-1)n}{2} \geq 2021$. Note that $2016 = 63 \cdot 32 = \frac{(64-1)\cdot 64}{2}$, so $n > 64$. With $n = 65$, we have $\frac{(n-1)n}{2} = \frac{64 \cdot 65}{2} = 32 \cdot 65 = 2016 + 64 \geq 2021$, so the answer is $n = \boxed{65}$.

3. First, we need f to be continuous at c, which requires $(c-a)^2 + b = c^3 - c$. Second, we need f' to be continuous at c. Since
$$f'(x) = \begin{cases} 2(x-a) & \text{if } x > c \\ 3x^2 - 1 & \text{if } x < c, \end{cases}$$
this means that both $2(x-a)$ and $3x^2 - 1$ must converge to the same value as x approaches c, which means $2(c-a) = 3c^2 - 1$. Lastly, we need f'' to be continuous at c. Since
$$f''(x) = \begin{cases} 2 & \text{if } x > c \\ 6x & \text{if } x < c, \end{cases}$$
this means that $6x$ must converge to 2 as x approaches c, which means $6c = 2$. This last equation tells us $c = \frac{1}{3}$, so
$$2(c-a) = 3c^2 - 1 \implies 2\left(\frac{1}{3} - a\right) = \frac{3}{9} - 1 \implies \frac{2}{3} - 2a = -\frac{2}{3} \implies a = \frac{2}{3}.$$
This further means
$$(c-a)^2 + b = c^3 - c \implies \left(\frac{1}{3} - \frac{2}{3}\right)^2 + b = \frac{1}{27} - \frac{1}{3} \implies b = -\frac{8}{27} - \frac{1}{9} = -\frac{11}{27}.$$
Hence, $\frac{p}{q} = a + b + c = \frac{2}{3} - \frac{11}{27} + \frac{1}{3} = \frac{16}{27}$, so $p + q = 16 + 27 = \boxed{43}$.

4. We use implicit differentiation. The equation $f(x) + (f(x))^3 = x + x^7$ holds for all $x \in \mathbb{R}$, so both sides of this equation have the same derivative with respect to x. By the Chain Rule, this means
$$f'(x) + 3(f(x))^2 \cdot f'(x) = 1 + 7x^6 \implies \left(1 + 3(f(x))^2\right) f'(x) = 1 + 7x^6 \implies f'(x) = \frac{1 + 7x^6}{1 + 3(f(x))^2}.$$
Then,
$$f'(2) = \frac{1 + 7 \cdot 64}{1 + 3(f(2))^2} = \frac{449}{1 + 3(f(2))^2}.$$
We also know $f(2) + (f(2))^3 = 2 + 2^7 = 130$, so $y = f(2)$ is a root to the polynomial $y^3 + y - 130 = 0$. This polynomial can be factored as $(y-5)(y^2 + 5y + 26) = 0$. Since $y^2 + 5y + 26 = (y + 5/2)^2 + 79/4 > 0$ for all real y, the only real solution of $y^3 + y - 130 = 0$ is $y = 5$, so $f(2) = 5$. Hence,
$$\frac{a}{b} = f'(2) = \frac{449}{1 + 3 \cdot 5^2} = \frac{449}{76}.$$
Since 449 and 76 are relatively prime, the answer is $a + b = 449 + 76 = \boxed{525}$.

5. The area of a triangle is half the product of its base length and its height, where the height is the distance between the base line and the opposing vertex. If x is 5 or 8, then T_x is a degenerate triangle and has area zero, so we may ignore these cases. Otherwise, no matter the value of x in the open interval $(5, 8)$, the triangle T_x has a fixed base between the fixed points $(5, 5^3)$ and $(8, 8^3)$; only the vertex opposing this fixed base of the triangle is allowed to vary. Therefore, maximizing the area of T_x is equivalent to maximizing the distance from the point (x, x^3) to the base. Since the point (x, x^3) is a continuously differentiable function of x, its distance from the base is also a continuously differentiable function of x, so this distance is maximized at a given point $x = \sqrt{n}$ if and only if the line tangent to the function $y = x^3$ at $x = \sqrt{n}$ is parallel to the base line connecting the points $(5, 5^3)$ and $(8, 8^3)$. The slope of this base line is $\frac{8^3-5^3}{8-5} = 8^2 + 8 \cdot 5 + 5^2 = 64 + 40 + 25 = 129$, and the slope of the tangent line of the function $y = x^3$ at $x = \sqrt{n}$ is simply the derivative of this function evaluated at $x = \sqrt{n}$, which equals $3\sqrt{n}^2$, or $3n$. Hence, $3n = 129$, so $n = \boxed{43}$.

6. Let $g(x) = xf(x)$ so that $g(x) - g(y) \le xy(x-y)^2 \le 400(x-y)^2$ for all x and y in $[1, 20]$. We claim that g is differentiable and its derivative is zero everywhere. The inequality $g(x) - g(y) \le 400(x-y)^2$ implies $\left|\frac{g(x)-g(y)}{x-y}\right| \le 400|x-y|$. Since absolute values are always nonnegative, the Squeeze theorem implies

$$0 \le \lim_{y\to x}\left|\frac{g(y)-g(x)}{y-x}\right| \le \lim_{y\to x} 400|y-x| = 400|x-x| = 0 \implies |g'(x)| = \lim_{y\to x}\left|\frac{g(y)-g(x)}{y-x}\right| = 0.$$

Now that we know g' is identically zero, it is easy to confirm that g must be a constant function (this follows from the Fundamental Theorem of Calculus). Therefore, $xf(x)$ equals some constant k for all $x \in [1, 20]$, so $f(x) = k/x$. In order for the range of f to be a subset of $[-100, 100]$, we need $|f(1)| \le 100$, so $|k| \le 100$. Regardless of whether k is positive or negative, f is a monotonic function, so $f(x) - f(y)$ is maximized when x and y are 1 and 20 in some order, and the maximum of this value, for a given constant k, is $|k|\left(\frac{1}{1} - \frac{1}{20}\right) = \frac{19|k|}{20}$. To maximize this quantity over all valid k, we pick $k \in \{-100, 100\}$, so that the maximum possible value of $f(x) - f(y)$ is $\frac{19\cdot 100}{20} = \boxed{95}$.

7. Let $\mathcal{B}$ denote the solid of interest. For any given $z \in [0, 900]$, let $\mathcal{S}_z$ be the set of all points (x, y, z) in $\mathcal{B}$. $\mathcal{S}_z$ is congruent to the compression of the surface $\mathcal{S}$ by a scaling factor of $\frac{900-z}{900}$. Thus, if we use the notation $[K]$ to denote the area of an arbitrary flat surface K, then

$$[\mathcal{S}_z] = \left(\frac{900-z}{900}\right)^2 [\mathcal{S}].$$

We can now compute the volume of the solid $\mathcal{B}$ as

$$\int_0^{900} [\mathcal{S}_z]\, dz = [\mathcal{S}] \int_0^{900} \left(\frac{900-z}{900}\right)^2 dz = 900[\mathcal{S}] \int_0^1 \left(\frac{900-900u}{900}\right)^2 du = 900[\mathcal{S}] \int_0^1 (1-u)^2\, du,$$

where we used the substitution $z = 900u$. Note that $\int_0^1 (1-u)^2\, du = -\frac{(1-u)^3}{3}\Big|_{u=0}^{u=1} = \frac{1}{3}$ and $[\mathcal{S}] = \int_{-1}^1 (2 - (x^2+1))\, dx = \int_{-1}^1 (1-x^2)\, dx = 2 - \frac{2}{3} = \frac{4}{3}$. Thus, the volume of $\mathcal{B}$ is $900 \cdot \frac{4}{3} \cdot \frac{1}{3} = \boxed{400}$.

8. We have the indefinite integral

$$\int \left(\frac{1}{\ln x} - \frac{2}{(\ln x)^3}\right) dx = \frac{x}{(\ln x)^2} + \frac{x}{\ln x} + C,$$

so for real numbers $a > 1$,

$$\begin{aligned}\int_a^{a^2} \left(\frac{1}{\ln x} - \frac{2}{(\ln x)^3}\right) dx &= \frac{a^2}{(\ln(a^2))^2} + \frac{a^2}{\ln(a^2)} - \frac{a}{(\ln a)^2} - \frac{a}{\ln a} = \frac{a^2}{4(\ln a)^2} + \frac{a^2}{2\ln a} - \frac{a}{(\ln a)^2} - \frac{a}{\ln a} \\ &= \left(\frac{a}{4} - 1\right)\frac{a}{(\ln a)^2} + \left(\frac{a}{2} - 1\right)\frac{a}{\ln a}.\end{aligned}$$

If the above quantity equals $\frac{a}{\ln a}$, then

$$0 = \left(\frac{a}{4} - 1\right)\frac{a}{(\ln a)^2} + \left(\frac{a}{2} - 1\right)\frac{a}{\ln a} - \frac{a}{\ln a} = \left(\frac{a}{4} - 1\right)\frac{a}{(\ln a)^2} + \left(\frac{a}{2} - 2\right)\frac{a}{\ln a}$$

$$= \left(\frac{a}{4} - 1\right)\left(\frac{a}{(\ln a)^2} + \frac{2a}{\ln a}\right) = \left(\frac{a}{4} - 1\right)\frac{a}{\ln a}\left(\frac{1}{\ln a} + 2\right).$$

Since $a > 1$, $\frac{a}{\ln a}$ and $\frac{1}{\ln a} + 2$ are strictly positive, so $\frac{a}{4} - 1$ must be zero. Thus, $a = \boxed{4}$.

9. Let $k = 2021/14$. Observe the following Product-to-Sum Identity:

$$\begin{aligned}\sin(a_{n-1}x)\sin(a_n x) &= \frac{1}{2}[(\cos(a_{n-1}x)\cos(a_n x) + \sin(a_{n-1}x)\sin(a_n x)) \\ &\qquad - (\cos(a_{n-1}x)\cos(a_n x) - \sin(a_{n-1}x)\sin(a_n x))] \\ &= \frac{1}{2}\left(\cos((a_n - a_{n-1})x) - \cos((a_n + a_{n-1})x)\right),\end{aligned}$$

so

$$\begin{aligned}\int_0^{k\pi} \sin(a_{n-1}x)\sin(a_n x)\,dx &= \frac{1}{2}\int_0^{k\pi} \left(\cos((a_n - a_{n-1})x) - \cos((a_n + a_{n-1})x)\right)\,dx \\ &= \frac{1}{2}\left(\frac{\sin((a_n - a_{n-1})x)}{a_n - a_{n-1}} - \frac{\sin((a_n + a_{n-1})x)}{a_n + a_{n-1}}\right)\Bigg|_{x=0}^{x=k\pi} \\ &= \frac{1}{2}\left(\frac{\sin(k(a_n - a_{n-1})\pi)}{a_n - a_{n-1}} - \frac{\sin(k(a_n + a_{n-1})\pi)}{a_n + a_{n-1}}\right).\end{aligned}$$

From the recurrence relation $a_n = 2a_{n-1} + a_{n-2}$ for $n \geq 2$, we have $a_n - a_{n-1} = a_{n-1} + a_{n-2}$, so for all $n \geq 2$,

$$\int_0^{k\pi} \sin(a_{n-1}x)\sin(a_n x)\,dx = \frac{1}{2}\left(\frac{\sin(k(a_{n-1} + a_{n-2})\pi)}{a_{n-1} + a_{n-2}} - \frac{\sin(k(a_n + a_{n-1})\pi)}{a_n + a_{n-1}}\right).$$

Thus, we have the telescoping sum

$$\begin{aligned}\sum_{n=2}^{\infty}\int_0^{k\pi} \sin(a_{n-1}x)\sin(a_n x)\,dx &= \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{\sin(k(a_{n-1} + a_{n-2})\pi)}{a_{n-1} + a_{n-2}} - \frac{\sin(k(a_n + a_{n-1})\pi)}{a_n + a_{n-1}}\right) \\ &= \frac{1}{2}\left(\frac{\sin(k(a_1 + a_0)\pi)}{a_1 + a_0} - \lim_{n\to\infty}\frac{\sin(k(a_n + a_{n-1})\pi)}{a_n + a_{n-1}}\right) = \frac{1}{2}\cdot\frac{\sin(k(a_1 + a_0)\pi)}{a_1 + a_0}.\end{aligned}$$

Then,

$$\begin{aligned}\sum_{n=1}^{\infty}\int_0^{k\pi} \sin(a_{n-1}x)\sin(a_n x)\,dx &= \frac{1}{2}\left(\frac{\sin(k(a_1 - a_0)\pi)}{a_1 - a_0} - \frac{\sin(k(a_1 + a_0)\pi)}{a_1 + a_0}\right) + \frac{1}{2}\cdot\frac{\sin(k(a_1 + a_0)\pi)}{a_1 + a_0} \\ &= \frac{1}{2}\cdot\frac{\sin(k(a_1 - a_0)\pi)}{a_1 - a_0}.\end{aligned}$$

We now plug in the values $a_0 = 1$, $a_1 = 8$, and $k = 2021/14$:

$$\frac{1}{2}\cdot\frac{\sin(k(a_1 - a_0)\pi)}{a_1 - a_0} = \frac{\sin(2021(8-1)\pi/14)}{2(8-1)} = \frac{\sin(2021\pi/2)}{14} = \frac{\sin(1010\pi + \pi/2)}{14} = \frac{1}{14}.$$

Therefore, $p = 1$ and $q = 14$, so the answer is $p + q = \boxed{15}$.

10. We can write $P(x)$ in the form $a\prod_{j=1}^{d}(x-r_j)$ for some scalar a. For any x that is not a root of P,

$$\frac{1}{x-r_k}=\frac{a\prod_{j\in\{1,\ldots,d\}\setminus\{k\}}(x-r_j)}{a\prod_{j=1}^{d}(x-r_j)} \implies \sum_{k=1}^{d}\frac{1}{x-r_k}=\frac{\sum_{k=1}^{d}a\prod_{j\in\{1,\ldots,d\}\setminus\{k\}}(x-r_j)}{a\prod_{j=1}^{d}(x-r_j)}=\frac{P'(x)}{P(x)},$$

where we used the product rule to recognize the expression for $P'(x)=\frac{d}{dx}a\prod_{k=1}^{d}(x-r_k)$. Using partial fractions, we write

$$\sum_{k=1}^{d}\frac{1}{4-r_k^2}=\frac{1}{4}\sum_{k=1}^{d}\left(\frac{1}{2-r_k}+\frac{1}{2+r_k}\right)=\frac{1}{4}\left(\sum_{k=1}^{d}\frac{1}{2-r_k}-\sum_{k=1}^{d}\frac{1}{-2-r_k}\right).$$

Therefore, the desired sum equals $\frac{1}{4}\left(\frac{P'(2)}{P(2)}-\frac{P'(-2)}{P(-2)}\right)=\frac{1}{4}\left(\frac{3\cdot2^2+20\cdot2-1}{2^3+10\cdot2^2-12}-\frac{3\cdot(-2)^2+20\cdot(-2)-1}{(-2)^3+10\cdot(-2)^2-12}\right)=\frac{1}{4}\left(\frac{51}{36}-\frac{-29}{20}\right)=\frac{1}{16}\left(\frac{17}{3}+\frac{29}{5}\right)=\frac{172}{16\cdot15}=\frac{43}{60}$, so the answer is $43+60=\boxed{103}$.