Johns Hopkins Math Tournament 2021

Individual Round: Algebra and Number Theory

April 3rd, 2021

Instructions

- **Remember you must be proctored while taking the exam.**
- This test contains 10 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No outside help is allowed. This includes people, the internet, translators, books, notes, calculators, or any other computational aid. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor.
- Good luck!

1. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Find the value of the sum

$$\left\lfloor 2+\frac{1}{2^{2021}}\right\rfloor+\left\lfloor 2+\frac{1}{2^{2020}}\right\rfloor+\cdots+\left\lfloor 2+\frac{1}{2^{1}}\right\rfloor+\left\lfloor 2+\frac{1}{2^{0}}\right\rfloor.$$

2. David has some pennies. One apple costs 3 pennies, one banana costs 5 pennies, and one cranberry costs 7 pennies. If David spends all his money on apples, he will have 2 pennies left; if David spends all his money on bananas, he will have 3 pennies left; is David spends all his money on cranberries, he will have 2 pennies left. What is the least possible amount of pennies that David can originally have?

3. Let $B = \{2^1, 2^2, 2^3, \dots, 2^{21}\}$. Find the remainder when

$$\sum_{m,n\in B:\ m<n} \gcd(m,n)$$

is divided by 1000, where the sum is taken over all pairs of elements (m,n) of B such that $m<n$.

4. For a natural number n, let a_n be the sum of all products xy over all integers x and y with $1 \le x < y \le n$. For example, $a_3 = 1\cdot 2 + 2\cdot 3 + 1\cdot 3 = 11$. Determine the smallest $n \in \mathbb{N}$ such that $n > 1$ and a_n is a multiple of 2020.

5. A function f with domain A and range B is called *injective* if every input in A maps to a unique output in B (equivalently, if $x, y \in A$ and $x \neq y$, then $f(x) \neq f(y)$). With $\mathbb{C}$ denoting the set of complex numbers, let P be an injective polynomial with domain and range $\mathbb{C}$. Suppose further that $P(0) = 2021$ and that when P is written in standard form, all coefficients of P are integers. Compute the smallest possible positive integer value of $P(10)/P(1)$.

6. A sequence of positive integers $\{a_0, a_1, a_2, \dots\}$ satisfies $a_0 = 83$ and $a_n = (a_{n-1})^{a_{n-1}}$ for all positive integers n. Compute the remainder when a_{2021} is divided by 60.

7. A line passing through $(20, 21)$ intersects the curve $y = x^3 - 2x^2 - 3x + 5$ at three distinct points A, B, and C, such that B is the midpoint of $\overline{AC}$. The slope of this line is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

8. For complex number constant c, and real number constants p and q, there exist three distinct complex values of x that satisfy $x^3 + cx + p(1+qi) = 0$. Suppose c, p, and q were chosen so that all three complex roots x satisfy $\frac{5}{6} \le \frac{\mathrm{Im}(x)}{\mathrm{Re}(x)} \le \frac{6}{5}$, where $\mathrm{Im}(x)$ and $\mathrm{Re}(x)$ are the imaginary and real part of x, respectively. The largest possible value of $|q|$ can be expressed as a common fraction $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

9. Let a and b be positive real numbers such that $\log_{43} a = \log_{47}(3a+4b) = \log_{2021} b^2$. Then, the value of $\frac{b^2}{a^2}$ can be written as $m + \sqrt{n}$, where m and n are integers. Find $m+n$.

10. A sequence of real numbers $\{a_1, a_2, a_3, \dots\}$ satisfies $0 \le a_1 \le 1$ and $a_{n+1} = \frac{1+\sqrt{a_n}}{2}$ for all positive integers n. If $a_1 + a_{2021} = 1$, then the product $a_1 a_2 a_3 \cdots a_{2020}$ can be written in the form m^k, where k is an integer and m is a positive integer that is not divisible by any perfect square greater than 1. Compute $m+k$.

Algebra and Number Theory Solutions

1. Notice that any fraction of the form $\frac{1}{2^n}$ for $n > 0$ will be strictly between 0 and 1, which means every term in the summation except for the last term will be strictly between 2 and 3. Thus, each of the first 2021 terms will evaluate to 2, and only the last term will evaluate to 3. The answer is $2021 \times 2 + 3 = \boxed{4045}$.

2. $\boxed{23}$. We will solve this problem using the Chinese remainder theorem. Thus, from the information given in the problem, if x is the number of pennies that David has, x satisfies the following equations:

$$x \equiv 2 \mod 3$$
$$x \equiv 3 \mod 5$$
$$x \equiv 2 \mod 7$$

From the first equation, we know that $x = 3a + 2$ for some integer a. Thus, we substitute this into the second equation to find:

$$3a + 2 \equiv 3 \mod 5$$
$$3a \equiv 1 \mod 5$$
$$a \equiv 2 \mod 5$$

From the equation above, we know that $a = 5b + 2$ for some integer b, and thus we substitute this back into the equation $x = 3a + 2$ above to find that $x = 15b + 8$. Now, we plug this equation for x into the third equation we formed from the problem, to find:

$$15b + 8 \equiv 2 \mod 7$$
$$15b \equiv 1 \mod 7$$
$$b \equiv 1 \mod 7$$

Thus, we find that $b = 7c + 1$ for some integer c, and thus we substitute this back into the equation $x = 15b + 8$ from above to conclude that $x = 105c + 23$. Since we are looking for the smallest positive value of x, we plug in $c = 0$ to find $x = 23$.

3. Observe that if $m = 2^i$ for some i, then

$$\sum_{n \in B:\ n > 2^i} \gcd(2^i, n) = \sum_{n \in B:\ n > 2^i} 2^i,$$

as n is a power of 2 greater than 2^i. Now, if we write $n = 2^j$ for some j, notice that there are $21 - i$ possible values for j (since we want $2^j > 2^i$), and hence $21 - i$ possible n. Thus, we can evaluate the above sum to be $(21 - i)2^i$. We know that i ranges from 1 to 21, so it suffices to evaluate

$$S = \sum_{i=1}^{21} (21 - i)2^i = 20(2^1) + 19(2^2) + 18(2^3) + \cdots + 2(2^{19}) + 1(2^{20}).$$

We have that

$$2S = 20(2^2) + 19(2^3) + 18(2^4) + \cdots + 2(2^{20}) + 1(2^{21}),$$

so

$$\begin{aligned} S &= 2S - S \\ &= 20(2^2) + 19(2^3) + 18(2^4) + \cdots + 2(2^{20}) + 1(2^{21}) \\ &\quad - 20(2^1) - 19(2^2) - 18(2^3) - \cdots - 2(2^{19}) - 1(2^{20}) \\ &= 2^2 + 2^3 + \cdots + 2^{20} + 2^{21} - 20(2) \\ &= 2^2(1 + 2 + \cdots + 2^{18} + 2^{19}) - 20(2) \\ &= 2^2(2^{20} - 1) - 20(2) \\ &= 2^{22} - 44. \end{aligned}$$

Lastly, note that $2^{22} = (2^{10})^2 \cdot 2^2$, so $2^{22} \equiv 24^2 \cdot 4 \equiv 304 \pmod{1000}$. The answer is then $304 - 44 = \boxed{260}$.

4. The expansion of $(1 + 2 + \cdots + n)^2$ includes terms of a_n:

$$\left(\sum_{k=1}^{n} k\right)^2 = \sum_{k=1}^{n} k^2 + 2\sum_{x=1}^{n-1}\sum_{y=x+1}^{n} xy = \left(\sum_{k=1}^{n} k^2\right) + 2a_n.$$

Recall the formulas $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. Now, we can see that

$$a_n = \frac{1}{2}\left(\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right) = \frac{n(n+1)(3n(n+1) - 2(2n+1))}{24} = \frac{n(n+1)(3n+2)(n-1)}{24}.$$

Because $24 = 2^3 \cdot 3$ and $2020 = 2^2 \cdot 5 \cdot 101$, a_n is divisible by 2020 if and only if $b_n := n(n+1)(3n+2)(n-1)$ is divisible by $24 \cdot 2020 = 2^5 \cdot 3 \cdot 5 \cdot 101$. Note that one of n, $n+1$, or $n-1$ must be a multiple of 3; thus, 3 always divides b_n. Also note that, of the four factors n, $n+1$, $3n+2$, and $n-1$, it will always be the case that two factors are odd and two factors are even, with exactly one of those even factors divisible by 4. Thus, in order for 2^5 to divide b_n, one of the four factors in b_n must be a multiple of 2^4 (or 16). This means that $n \bmod 16 \in \{0, 1, 10, 15\}$ (the 10 comes from the fact that $3 \cdot (10) + 2$ is a multiple of 16). For b_n to be divisible by 5, n needs to be equivalent to 0, 1, or 4 modulo 5. Lastly, for b_n to be divisible by 101, n must be equivalent to 0, 1, 33, or 100 modulo 101. This last condition is the most restrictive since there are way more integral residues modulo 101 than modulo 5 or 16. Dropping $n = 0$ and $n = 1$, we list the first few positive integers that are 0, 1, 33, or 100 modulo 101: $\{33, 100, 101, 102, 134, 201, 202, 203, 235, 302, 303, 304, 336, 403, \ldots\}$. It is easy to pinpoint and remove values equivalent to 2 or 3 modulo 5; doing so leaves $\{100, 101, 134, 201, 235, 304, 336, \ldots\}$. The smallest remaining value with a desirable residue modulo 16 is $n = 304$, which is itself a multiple of 16. Hence, $\boxed{304}$ is the smallest integer value of n greater than 1 such that $2020 \mid a_n$.

5. Every non-constant polynomial with complex coefficients has at least one complex root. If our polynomial P is injective, then it cannot be constant, so for any complex constant k, all roots x to the equation $P(x) - k = 0$ must be the same complex number. For a specific choice of k, suppose the unique root to the equation is r, and let d be the degree of P. Then, the equation $P(x) - k = 0$ must simplify to $c(x - r)^d = 0$ for some nonzero coefficient c, so $P(x) = c(x - r)^d + k$. Suppose k was chosen to be nonzero. Then, the equation $P(x) = 0$ reduces to $c(x - r)^d = -k$, which has d distinct solutions x. Hence, P has d distinct roots, so P cannot be injective if $d > 1$. On the contrary, if $d = 1$, then P is a linear equation and must be injective, so the polynomial P is injective if and only if the degree of P is 1.

We may now parameterize P by $P(x) = 2021 + ax$ because we are given $P(0) = 2021$. We are also given that a must be an integer, and since P needs to be injective, a cannot be zero. We have $\frac{P(y)}{P(x)} = \frac{2021+ay}{2021+ax}$. Note that $(2021 + ax) \mid (2021 + ay)$ holds if and only if $(2021 + ax) \mid a(y - x)$.

Under the assumption that $P(10)/P(1)$ is an integer, this divisibility condition must hold when $x = 1$ and $y = 10$, so $(2021 + a) \mid 9a$, so $(2021 + a)k = 9a$ for some integer k. Since this k satisfies

$k+1 = \frac{9a}{2021+a} + 1 = \frac{2021+10a}{2021+a} = \frac{P(10)}{P(1)}$, the assumption that $P(10)/P(1)$ is a *positive* integer means that k is nonnegative. Thus, k is a nonnegative integer satisfying $(2021+a)k = 9a$, or

$$(a+2021)(k-9) = -9 \cdot 2021.$$

Since a is allowed to be any integer except for zero, we now just need to find the minimum nonnegative integer value of k such that $a = \frac{-9\cdot 2021}{k-9} - 2021$ is a nonzero integer. Importantly, $k = 0$ is disallowed because this would produce $a = 2021 - 2021 = 0$, which is disallowed. Thus, k must be positive, so $k-9 \geq -8$. The smallest integer that is at least -8 and also a factor of $-9\cdot 2021$ is -3, so the minimum value of k satisfies $k-9 = -3$, i.e., $k = 6$. Hence, the smallest positive integer value of $P(10)/P(1)$ is $k+1 = 6+1 = \boxed{7}$.

6. Euler's Totient Theorem states that $a^{\phi(m)} \equiv 1 \pmod m$ for any integer $m \geq 2$ and any integer a coprime to m, where $\phi(m)$ is the number of integers in $\{1, 2, \ldots, m\}$ that are coprime to m. Since $\phi(60) = 16$, we observe that $a^a \equiv (a \bmod 60)^{(a \bmod 16)} \pmod{60}$ for any integer a coprime to 60. Similarly, since $\phi(16) = 8$, we have $a^a \equiv (a \bmod 16)^{(a \bmod 8)} \equiv (a \bmod 16)^{((a \bmod 16) \bmod 8)} \pmod{16}$. Thus, if we let $b_n = a_n \bmod 60$ and $c_n = a_n \bmod 16$ for all nonnegative integers n, then $b_0 = 23$, $c_0 = 3$, and $b_n = (b_{n-1})^{c_{n-1}} \bmod 60$ and $c_n = (c_{n-1})^{(c_{n-1} \bmod 8)} \bmod 16$ for all positive integers n. With this recurrence,

$$b_1 = 23^3 \bmod 60 = ((23^2 \bmod 60)\cdot 23) \bmod 60 = (-11\cdot 23) \bmod 60 = (-253) \bmod 60 = 47,$$

$$c_1 = 3^3 \bmod 16 = 27 \bmod 16 = 11,$$

$$\begin{aligned} b_2 &= 47^{11} \bmod 60 = (-13^{11}) \bmod 60 = (-(13^2)^5 \cdot 13) \bmod 60 = (-(-11)^5 \cdot 13) \bmod 60 \\ &= ((11^2)^2 \cdot 11 \cdot 13) \bmod 60 = ((1)^2 \cdot 11 \cdot 13) \bmod 60 = 143 \bmod 60 = 23, \end{aligned}$$

and

$$c_2 = 11^{(11 \bmod 8)} \bmod 16 = 11^3 \bmod 16 = (-5)^3 \bmod 16 = (-125) \bmod 16 = 3.$$

Since $(b_2, c_2) = (b_0, c_0) = (23, 3)$, we conclude that $(b_{2k}, c_{2k}) = (23, 3)$ and $(b_{2k+1}, c_{2k+1}) = (47, 11)$ for all nonnegative integers k. Therefore, the answer to the problem is $a_{2021} \bmod 60 = b_{2021} = \boxed{47}$.

7. Let

$$f(x) = x^3 - 2x^2 - 3x + 5$$

and let the line have equation

$$g(x) = mx + b.$$

Since $g(x)$ intersects $f(x)$ at 3 equally spaced points, it is easy to see that the polynomial $f(x) - g(x)$ has 3 distinct real roots, and these roots, say r_1, r_2, and r_3, form an arithmetic progression. Without loss of generality, assume that $r_1 < r_2 < r_3$. Now, we let

$$\begin{aligned} r_1 &= a - d \\ r_2 &= a \\ r_3 &= a + d \end{aligned}$$

for real numbers a and d. Then, we have

$$f(x) - g(x) = (x-a+d)(x-a)(x-a-d).$$

Note that

$$f(x+a) - g(x+a) = x(x-d)(x+d),$$

so for simplicity, let $p(x) = f(x+a)$ and let $q(x) = g(x+a)$. We have

$$p(x) - q(x) = x(x^2 - d^2) = x^3 - d^2 x.$$

Observe that $p(x) - q(x)$ has no quadratic term and no constant term. Since $q(x)$ is linear, it cannot generate a quadratic term. Thus,

$$p(x) = f(x+a) = (x+a)^3 - 2(x+a)^2 - 3(x+a) + 5$$

needs to have no quadratic term. We can only generate quadratic terms from the cubic and quadratic terms from the cubic in $x + a$ above. In particular,

$$(x+a)^3 \implies 3ax^2$$

and

$$-2(x+a)^2 \implies -2x^2,$$

so

$$3ax^2 - 2x^2 = 0 \implies a = \frac{2}{3}.$$

Hence, $p(x) = f(x + \frac{2}{3})$, so

$$p(x) = \left(x + \frac{2}{3}\right)^3 - 2\left(x + \frac{2}{3}\right)^2 - 3\left(x + \frac{2}{3}\right) + 5 = x^3 - \frac{13}{3}x + \frac{65}{27}.$$

Moreover, $q(x) = g(x + \frac{2}{3})$, so

$$p(x) - q(x) = x^3 - \frac{13}{3}x + \frac{65}{27} - \left(m\left(x + \frac{2}{3}\right) + b\right) = x^3 - \left(m + \frac{13}{3}\right)x + \left(\frac{65}{27} - b - \frac{2}{3}m\right).$$

We seek the value of m. Since $p(x) - q(x)$ has no constant term, the above cubic must also not have one. Therefore,

$$b + \frac{2}{3}m = \frac{65}{27} \implies 27b + 18m = 65.$$

Since the line passes through $(20, 21)$, we have that

$$20m + b = 21.$$

Solving this system of equations in m and b, we obtain $m = \frac{251}{261}$, so the requested sum is $251 + 261 = \boxed{512}$.

8. Every complex number $z = a+bi$ can be written in polar form as $re^{i\theta}$. In this form, r is called the *modulus* of z and also equals $\sqrt{a^2+b^2}$, and θ is called the *argument* of z and satisfies $\theta \equiv \tan^{-1}\left(\frac{b}{a}\right) \pmod{\pi}$. Since $b = \mathrm{Im}(z)$ and $a = \mathrm{Re}(z)$, note that $\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} = \tan(\arg(z))$. Let z_1, z_2, and z_3 be the three complex roots to $x^3+cx+p(1+qi) = 0$. Then, $x^3+cx+p(1+qi) = (x-z_1)(x-z_2)(x-z_3)$, so $z_1z_2z_3 = -p(1+qi)$ and thus $q = -\frac{\mathrm{Im}(z_1z_2z_3)}{\mathrm{Re}(z_1z_2z_3)} = -\tan(\arg(z_1z_2z_3))$. We are told that $\frac{\mathrm{Im}(z_1)}{\mathrm{Re}(z_1)}, \frac{\mathrm{Im}(z_2)}{\mathrm{Re}(z_2)}, \frac{\mathrm{Im}(z_3)}{\mathrm{Re}(z_3)} \in [\frac{5}{6}, \frac{6}{5}]$. If we let $\theta_k = \tan^{-1}\frac{\mathrm{Im}(z_k)}{\mathrm{Re}(z_k)}$, then $\theta_k \equiv \arg(z_k) \pmod{\pi}$ for $k \in \{1,2,3\}$, so $\arg(z_1z_2z_3) \equiv \theta_1+\theta_2+\theta_3 \pmod{\pi}$. Because $q = -\tan(\theta_1+\theta_2+\theta_3)$, we maximize $|q|$ by making $\theta_1+\theta_2+\theta_3$ as close as possible to an odd integer multiple of $\pi/2$. Since $\theta_k \bmod \pi > \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$ and $\theta_k \bmod \pi < \tan^{-1}\sqrt{3} = \frac{\pi}{3}$, the value of $(\theta_1+\theta_2+\theta_3) \bmod \pi$ is strictly between $\pi/2$ and π, so we want to set each $(\theta_k \bmod \pi)$ to its minimum possible value, which is $\tan^{-1}\frac{5}{6}$. With $\theta \equiv \theta_1 \equiv \theta_2 \equiv \theta_3 \equiv \tan^{-1}\frac{5}{6} \pmod{\pi}$, the tangent triple-angle identity tells us

$$q = -\tan(3\theta) = -\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = -\frac{(5/6)(3 - 25/36)}{1 - 75/36} = \frac{415/216}{39/36} = \frac{415}{234}.$$

Since z_1, z_2, and z_3 are roots of the cubic polynomial $x^3 + cx + p(1+qi) = 0$, with a coefficient of 0 on the quadratic term, it remains to verify that we can choose the distinct roots z_1, z_2, and z_3 to satisfy Vieta's formula, $z_1 + z_2 + z_3 = 0$, as well as $\arg(z_k) \equiv \tan^{-1}\frac{5}{6} \pmod{\pi}$. Letting $\theta = \tan^{-1}\frac{5}{6}$, one way to do this is to choose $z_1 = e^{i\theta}$, $z_2 = 2e^{i\theta}$, and $z_3 = 3e^{i(\pi+\theta)}$. Thus, the absolute value of q is indeed maximized at $\frac{415}{234}$, so the answer is $415 + 234 = \boxed{649}$.

9. Using properties of logarithms, we can rewrite the given equations as

$$\begin{aligned}\frac{\log a}{\log 43} &= \frac{\log(3a+4b)}{\log 47}\\ \frac{\log(3a+4b)}{\log 47} &= \frac{2\log b}{\log 43+\log 47}\\ \frac{\log a}{\log 43} &= \frac{2\log b}{\log 43+\log 47}.\end{aligned}$$

Now, let

$$\begin{aligned}A &= \log a\\ B &= \log b\\ C &= \log(3a+4b)\\ D &= \log 43\\ E &= \log 47.\end{aligned}$$

We can now rewrite the system of equations as

$$\begin{aligned}\frac{A}{D} = \frac{C}{E} &\implies AE = CD\\ \frac{C}{E} = \frac{2B}{D+E} &\implies CD+CE = 2BE\\ \frac{A}{D} = \frac{2B}{D+E} &\implies AD+AE = 2BD.\end{aligned}$$

From the above, notice that

$$CD = 2BE - CE = E(2B-C) \implies \frac{C}{2B-C} = \frac{E}{D}$$

and

$$AE = 2BD - AD = D(2B-A) \implies \frac{2B-A}{A} = \frac{E}{D},$$

so

$$\frac{2B-A}{A} = \frac{C}{2B-C} \implies (2B-A)(2B-C) = AC \implies 4B^2-2BC-2AB=0 \implies 2B^2 = B(A+C).$$

We would like to divide the above equation by B, but we must first show that B cannot be zero (or equivalently, that $b \neq 1$). Suppose that $b = 1$. Then, the original equation from the problem statement tells us $\log_{43} a = \log_{2021} b^2 = 0$, so $a = 1$. But this means $0 = \log_{47}(3a+4b) = \log_{47} 7$, which contradicts the fact $\log_{47} 7 \neq 0$. Thus, we have that

$$2B = A+C \implies 2\log b = \log a + \log(3a+4b) \implies \log b^2 = \log(a(3a+4b)) \implies b^2 - 4ab - 3a^2 = 0.$$

Now, let $r = \frac{b}{a}$. We then have

$$r^2 - 4r - 3 = 0 \implies r = 2+\sqrt{7},$$

as a and b are positive real numbers. Therefore,

$$r^2 = \frac{b^2}{a^2} = (2+\sqrt{7})^2 = 11+4\sqrt{7} = 11+\sqrt{112}.$$

The requested sum is $11 + 112 = \boxed{123}$.

10. If $0 \le a_n \le 1$ for some n, then we may take the square root of both sides of the given equation to obtain $\sqrt{a_{n+1}} = \sqrt{\frac{1+\sqrt{a_n}}{2}}$, which resembles the half-angle cosine identity. Specifically, if we let $\sqrt{a_n} = \cos\alpha$ for some $\alpha \in [0, \pi/2]$, then $\sqrt{a_{n+1}} = \cos\frac{\alpha}{2}$. By a straightforward inductive argument, if we let $a_1 = \cos^2\theta$, then $a_n = \cos^2\left(\frac{\theta}{2^{n-1}}\right)$. The existence of such a θ in the interval $[0, \pi/2]$ is guaranteed by the given fact $0 \le a_1 \le 1$. We are also told

$$1 = a_1 + a_{2021} = \cos^2\theta + \cos^2\left(\frac{\theta}{2^{2020}}\right) = \cos^2\theta + \sin^2\left(\frac{\pi}{2} - \frac{\theta}{2^{2020}}\right).$$

The Pythagorean identity tells us $1 = \cos^2\theta + \sin^2\theta$, and since θ and $\frac{\pi}{2} - \frac{\theta}{2^{2020}}$ are both in $[0, \pi/2]$, we must have

$$\theta = \frac{\pi}{2} - \frac{\theta}{2^{2020}} \implies 2\theta = \pi - \frac{\theta}{2^{2019}} \implies \sin(2\theta) = \sin\left(\frac{\theta}{2^{2019}}\right).$$

Finally,

$$\prod_{n=1}^{2020} a_n = \left(\prod_{n=0}^{2019} \cos\frac{\theta}{2^n}\right)^2 = \left(\prod_{n=0}^{2019} \frac{\sin(\theta/2^{n-1})}{2\sin(\theta/2^n)}\right)^2 = \frac{\sin^2(2\theta)}{2^{4040}\sin^2(\theta/2^{2019})} = \frac{1}{2^{4040}} = 2^{-4040}.$$

Thus, $m = 2$ and $k = -4040$, so $m + k = \boxed{-4038}$.