Tiebreakers

2018

1. A large pond contains infinitely many lily pads labelled $1, 2, 3, \ldots$, placed in a line, where for each k, lily pad $k+1$ is one unit to the right of lily pad k. A frog starts at lily pad 100. Each minute, if the frog is at lily pad n, it hops to lily pad $n+1$ with probability $\frac{n-1}{n}$, and hops all the way back to lily pad 1 with probability $\frac{1}{n}$. Let N be the position of the frog after 1000 minutes. What is the expected value of N?

2. A cat is tied to one corner of the base of a tower. The base forms an equilateral triangle of side length 4 m, and the cat is tied with a leash of length 8 m. Let A be the area of the region accessible to the cat. If we write $A = \frac{m}{n}\pi + k\sqrt{\ell}$, where m, n, k, ℓ are positive integers such that m and n are relatively prime, and ℓ is squarefree, what is the value of $m+n+k+\ell$?

3. Compute
$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2+3n}-\frac{1}{n^2+3n+2}\right)$$

4. Find the sum of the real roots of $f(x) = x^4 + 9x^3 + 18x^2 + 18x + 4$.

5. Let a, b, c, d, e be the roots of $p(x) = 2x^5 - 3x^3 + 2x - 7$. Find the value of
$$(a^3-1)(b^3-1)(c^3-1)(d^3-1)(e^3-1)$$