Team Round

2018

1. Anita plays the following single-player game: She is given a circle in the plane. The center of this circle and some point on the circle are designated "known points". Now she makes a series of moves, each of which takes one of the following forms:

   (i) She draws a line (infinite in both directions) between two "known points"; or

   (ii) She draws a circle whose center is a "known point" and which intersects another "known point".

   Once she makes a move, all intersections between her new line/circle and existing lines/circles become "known points", unless the new/line circle is identical to an existing one. In other words, Anita is making a ruler-and-compass construction, starting from a circle.

   What is the smallest number of moves that Anita can use to construct a drawing containing an equilateral triangle inscribed in the original circle?

2. Compute the sum $\sum_{n=1}^{200} \frac{1}{n(n+1)(n+2)}$.

3. Let p be the third-smallest prime number greater than 5 such that:

   - $2p + 1$ is prime, and
   - $5^p \not\equiv 1 \pmod{2p+1}$.

   Find p.

4. If Percy rolls a fair six-sided die until he rolls a 5, what is his expected number of rolls, given that all of his rolls are prime?

5. Let $\triangle ABC$ be a right triangle such that $AB = 3$, $BC = 4$, $AC = 5$. Let point D be on AC such that the incircles of $\triangle ABD$ and $\triangle BCD$ are mutually tangent. Find the length of BD.

6. Karina has a polynomial $p_1(x) = x^2 + x + k$, where k is an integer. Noticing that p_1 has integer roots, she forms a new polynomial $p_2(x) = x^2 + a_1x + b_1$, where a_1 and b_1 are the roots of p_1 and $a_1 \geq b_1$. The polynomial p_2 also has integer roots, so she forms a new polynomial $p_3(x) = x^2 + a_2x + b_2$, where a_2 and b_2 are the roots of p_2 and $a_2 \geq b_2$. She continues this process until she reaches $p_7(x)$ and finds that it does not have integer roots. What is the largest possible value of k?

7. For a positive number n, let $g(n)$ be the product of all $1 \leq k \leq n$ such that $\gcd(k, n) = 1$, and say that $n > 1$ is *reckless* if n is odd and $g(n) \equiv -1 \pmod{n}$. Find the number of reckless numbers less than 50.

8. Find the largest positive integer n that cannot be written as $n = 20a + 28b + 35c$ for nonnegative integers a, b, and c.

9. Say that a function $f : \{1, 2, \ldots, 1001\} \to \mathbb{Z}$ is *almost polynomial* if there is a polynomial $p(x) = a_{200}x^{200} + \cdots + a_1 x + a_0$ such that each a_n is an integer with $|a_n| \leq 201$, and such that $|f(x) - p(x)| \leq 1$ for all $x \in \{1, 2, \ldots, 1001\}$. Let N be the number of almost polynomial functions. Compute the remainder upon dividing N by 199.

10. Let ABC be a triangle such that $AB = 13, BC = 14, AC = 15$. Let M be the midpoint of BC and define $P \neq B$ to be a point on the circumcircle of ABC such that $BP \perp PM$. Furthermore, let H be the orthocenter of ABM and define Q to be the intersection of BP and AC. If R is a point on HQ such that $RB \perp BC$, find the length of RB.