Individual Round

2018

1. Two robots race on the plane from $(0,0)$ to (a,b), where a and b are positive real numbers with $a<b$. The robots move at the same constant speed. However, the first robot can only travel in directions parallel to the lines $x=0$ or $y=0$, while the second robot can only travel in directions parallel to the lines $y=x$ or $y=-x$. Both robots take the shortest possible path to (a,b) and arrive at the same time. Find the ratio $\frac{a}{b}$.

2. Suppose $x+\frac{1}{x}+y+\frac{1}{y}=12$ and $x^2+\frac{1}{x^2}+y^2+\frac{1}{y^2}=70$. Compute $x^3+\frac{1}{x^3}+y^3+\frac{1}{y^3}$.

3. Find the largest non-negative integer a such that 2^a divides
$$3^{2^{2018}}+3.$$

4. Suppose z and w are complex numbers, and $|z|=|w|=z\bar{w}+\bar{z}w=1$. Find the largest possible value of $\mathrm{Re}(z+w)$, the real part of $z+w$.

5. Two people, A and B, are playing a game with three piles of matches. In this game, a *move* consists of a player taking a positive number of matches from one of the three piles such that the number remaining in the pile is equal to the nonnegative difference of the numbers of matches in the other two piles. A and B each take turns making moves, with A making the first move. The last player able to make a move wins. Suppose that the three piles have 10, x, and 30 matches. Find the largest value of x for which A does not have a winning strategy.

6. Let $A_1A_2A_3A_4A_5A_6$ be a regular hexagon with side length 1. For $n=1,\ldots,6$, let B_n be a point on the segment A_nA_{n+1} chosen at random (where indices are taken mod 6, so $A_7=A_1$). Find the expected area of the hexagon $B_1B_2B_3B_4B_5B_6$.

7. A termite sits at the point $(0,0,0)$, at the center of the octahedron $|x|+|y|+|z|\le 5$. The termite can only move a unit distance in either direction parallel to one of the x, y, or z axes: each step it takes moves it to an adjacent lattice point. How many distinct paths, consisting of 5 steps, can the termite use to reach the surface of the octahedron?

8. Let
$$P(x)=x^{4037}-3-8\cdot\sum_{n=1}^{2018}3^{n-1}x^n.$$
Find the number of roots z of $P(x)$ with $|z|>1$, counting multiplicity.

9. How many times does 01101 appear as a not necessarily contiguous substring of 0101010101010101? (Stated another way, how many ways can we choose digits from the second string, such that when read in order, these digits read 01101?)

10. A *perfect number* is a positive integer that is equal to the sum of its proper positive divisors, that is, the sum of its positive divisors excluding the number itself. For example, 28 is a perfect number because $1+2+4+7+14=28$. Let n_i denote the i^{th} smallest perfect number. Define

$$f(x)=\sum_{i|n_x}\sum_{j|n_i}\frac{1}{j}$$

(where $\sum_{i|n_x}$ means we sum over all positive integers i that are divisors of n_x). Compute $f(2)$, given there are at least 50 perfect numbers.

11. Let O be a circle with chord AB. The perpendicular bisector to AB is drawn, intersecting O at points C and D, and intersecting AB at the midpoint E. Finally, a circle O' with diameter ED is drawn, and intersects the chord AD at the point F. Given $EC=12$, and $EF=7$, compute the radius of O.

12. Suppose r,s,t are the roots of the polynomial x^3-2x+3. Find

$$\frac{1}{r^3-2}+\frac{1}{s^3-2}+\frac{1}{t^3-2}$$

13. Let $a_1,a_2,\ldots,a_{14}$ be points chosen independently at random from the interval $[0,1]$. For $k=1,2,\ldots,7$, let I_k be the closed interval lying between a_{2k-1} and a_{2k} (from the smaller to the larger). What is the probability that the intersection of $I_1,I_2,\ldots,I_7$ is nonempty?

14. Consider all triangles $\triangle ABC$ with area $144\sqrt{3}$ such that

$$\frac{\sin A\sin B\sin C}{\sin A+\sin B+\sin C}=\frac{1}{4}$$

Over all such triangles ABC, what is the smallest possible perimeter?

15. Let N be the number of sequences $(x_1,x_2,\ldots,x_{2018})$ of elements of $\{1,2,\ldots,2019\}$, not necessarily distinct, such that $x_1+x_2+\cdots+x_{2018}$ is divisible by 2018. Find the last three digits of N.